Solutions to the Algebra problems on the Comprehensive Examination of January 29, 2010

1. (10 Points) Let G and G' be groups, and let $\phi : G \to G'$ and $\psi : G \to G'$ be homomorphisms. Define
$$H = \{g \in G \mid \phi(g) = \psi(g)\}.$$
Prove that H is a subgroup of G.

   Solution: (H nonempty) Since ϕ and ψ are homomorphisms, $\phi(e_G) = \psi(e_G) = e_{G'}$, so $e_G \in H$, so $H \neq \emptyset$.✓
(H closed under group operation) Given $a, b \in H$, since ϕ and ψ are homomorphisms, $\phi(ab) = \phi(a)\phi(b) = \psi(a)\psi(b) = \psi(ab)$ so $ab \in H$.✓
(H closed under $^{-1}$) Given $a \in H$, since ϕ and ψ are homomorphisms, $\phi(a^{-1}) = (\phi(a))^{-1} = (\psi(a))^{-1} = \psi(a^{-1})$ so $a^{-1} \in H$.✓
Thus H is a subgroup of G as desired. QED

2. (10 Points) Let G be an abelian group. Let T be the set of elements of G that have finite order.

   (a) Show that T is a subgroup of G.

   Solution: (T nonempty) $o(e) = 1$ so $e \in T$, so $T \neq \emptyset$.✓
(T closed under group operation) Given $a, b \in T$, by rules of exponentiation (also note that $o(a)$ and $o(b)$ are finite):
$$(ab)^{o(a)o(b)} = (a^{o(a)})^{o(b)}(b^{o(b)})^{o(a)} = e^{o(b)}e^{o(a)} = e.$$
Thus $o(ab) \leq o(a)o(b)$ which means $o(ab)$ is finite, so $ab \in T$.✓
(T closed under $^{-1}$) Given $a \in T$, $o(a)$ is finite so $(a^{-1})^{o(a)} = (a^{o(a)})^{-1} = e^{-1} = e$. Thus $o(a^{-1}) \leq o(a)$ (in fact they are equal, but that is not important here) so $a^{-1} \in T$.✓
Thus T is a subgroup of G as desired. QED

   (b) Show that in G/T, the only element of finite order is the identity.

   Solution: Take $a \in G$ such that Ta has finite order $o(Ta) = m$. Then $T(a^m) = (Ta)^m = Te$, so $a^m = a^m e^{-1} \in T$. But then a^m has finite order. Let $n = o(a^m)$, so $a^{mn} = (a^m)^n = e$, so $o(a) \leq mn$, which means that a has finite order so $a \in T$. Then $Ta = Te = T$. Since T is the identity element of G/T, it follows that Ta is the identity element, as desired. QED

3. (10 Points) Let σ be the permutation $(4\ 2\ 1)(6\ 1\ 3\ 2)$ in S_6.

   (a) Write σ as a product of disjoint cycles in S_6.

   Solution: $\sigma = (1\ 3)(2\ 6\ 4)$

(b) Compute the **order** of σ.

Solution: The order of σ is the lcm of the orders of each individual cycle (and the order of an n-cycle is n): $o(\sigma) = \text{lcm}(3, 2) = 6$.

(c) Is σ an even or an odd permutation?

Solution: $\sigma = (1\ 3)(2\ 6)(6\ 4)$ which is the product of 3 transpositions so σ is odd.

4. (10 Points) Let R be a commutative ring and $S \subseteq R$ a subset of R. Define the *annihilator* of S in R to be

$$\text{Ann}(S) = \{r \in R \mid rs = 0 \text{ for every } s \in S\}$$

(a) Show that $\text{Ann}(S)$ is an ideal of R.

Solution: First, we show that $(\text{Ann}(S), +)$ is a subgroup of $(R, +)$:
($\text{Ann}(S)$ nonempty) Trivially $0 \in \text{Ann}(S)$ since $0s = 0\ \forall s \in S$, so $\text{Ann}(S) \neq \emptyset$.✓
($\text{Ann}(S)$ closed under $+$) Given $a, b \in \text{Ann}(S)$ and $s \in S$, $(a+b)s = as + bs = 0 + 0 = 0$, so $a + b \in \text{Ann}(S)$.✓
(I closed under negatives) Given $a \in \text{Ann}(S)$ and $s \in S$, $(-a)s = -(as) = -0 = 0$, so $-a \in \text{Ann}(S)$.✓
Thus $(\text{Ann}(S), +)$ is a subgroup of $(R, +)$.✓
Now given $r \in R$, $x \in \text{Ann}(S)$, $s \in S$, $(rx)s = r(xs) = r(0) = 0$ so $rx \in \text{Ann}(S)$.
Similarly (but this time using the commutativity of R), $(xr)s = (rx)s = r(xs) = r(0) = 0$ so $xr \in \text{Ann}(S)$.✓
Thus, I is an ideal of R. QED

(b) If S and T are both subsets of R, show that

$$\text{Ann}(S) \cap \text{Ann}(T) = \text{Ann}(S \cup T).$$

Solution: $\subseteq$:
Take $x \in \text{Ann}(S) \cap \text{Ann}(T)$ and $y \in S \cup T$, either $y \in S$ or $y \in T$. If $y \in S$, then $xy = 0$ because $x \in \text{Ann}(S)$ and if $y \in T$, then $xy = 0$ because $x \in \text{Ann}(T)$. Thus $xy = 0$ no matter what, so $x \in \text{Ann}(S \cup T)$, so $\text{Ann}(S) \cap \text{Ann}(T) \subseteq \text{Ann}(S \cup T)$.✓
$\supseteq$:
Take $x \in \text{Ann}(S \cup T)$:
Given $s \in S \subseteq (S \cup T)$, $xs = 0$ so $x \in \text{Ann}(S)$.✓
Given $t \in T \subseteq (S \cup T)$, $xt = 0$ so $x \in \text{Ann}(T)$.✓
Thus, $x \in \text{Ann}(S) \cap \text{Ann}(T)$, so $\text{Ann}(S) \cap \text{Ann}(T) \supseteq \text{Ann}(S \cup T)$.✓
Thus $\text{Ann}(S) \cap \text{Ann}(T) = \text{Ann}(S \cup T)$ as desired. QED