Math 28 Spring 2008: Final Exam

Instructions: This exam is to be turned in no later than Friday May 9th in class (10am). You may use your book and your notes from class, but no additional resources or discussion are permitted. If you need clarification on a problem, please ask me. Each problem will be graded out of 10 points for a total of 100 possible points.

Problem 1. Define $a_1 = 0, a_2 = \frac{1}{2}, a_{n+1} = \frac{1}{3}(1 + a_n + a_{n-1}^3)$. Show that the sequence (a_n) converges and find its value of convergence.

Proof. We will first show by induction that the sequence is increasing. We know $a_1 = 0, a_2 = \frac{1}{2}, a_3 = \frac{1}{2}$, so we have established the base case. Assume now that $a_{n-1} \geq a_{n-2}$ and $a_n \geq a_{n-1}$. Then,

$$a_{n+1} = \frac{1}{3}(a_n + a_{n-1}^3) \geq \frac{1}{3}(a_{n-1} + a_{n-2}^3) = a_n.$$

So we have established that (a_n) is increasing. To show that it is monontone, note that $a_1 = 0$ and the sequence is increasing so it is bounded below by 0. Now we will show by induction that it is bounded above by 1. The base case is established by $a_1 = 0, a_2 = \frac{1}{2}$. So in general

$$a_{n+1} = \frac{1}{3}(1 + a_n + a_{n-1}^3) \leq \frac{1}{3}(1 + 1+) = 1.$$

So the sequence is bounded and montone so it is convergent by the Monotone Convergence Theorem. Let a be the value of convergece of a. Then $(a_n) \to a$, $(a_{n-1}) \to a$, and $(a_{n-2}) \to a$ since all of these sequences are eventually the same. So we have

$$a = \frac{1}{3}(1 + a + a^3).$$

So in particular

$$a = 1, \frac{-1+\sqrt{5}}{2}, \text{ or } \frac{-1-\sqrt{5}}{2}$$

Note that the sequence starts at 0 and is increasing, so it converges to the smallest positive value. Since the third is negative and the second is smaller than 1 we have

$$a = \frac{-1+\sqrt{5}}{2}.$$

□

Problem 2. In this problem we consider a modified Harmonic Series. Let the (p, q)-Harmonic series, be the Harmonic series with p consecutive positive terms followed by q consecutive negative terms. In other words, the $(2, 3)$-Harmonic series is given by

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \cdots.$$

Show that the (p, q) Harmonic series converges if and only if $p = q$.

Proof. We first show that if $p = q$ then the series converges. We have

$$S_{lp} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{p}\right) - \left(\frac{1}{p+1} + \cdots + \frac{1}{2p}\right) + \cdots + (-1)^{l+1}\left(\frac{1}{(l-1)p+1} + \cdots + \frac{1}{lp}\right)$$

Therefore, S_{lp} is the partial sum of an alternating series. Since p is a constant we have

$$\left|\frac{1}{(l-1)p+1} + \cdots + \frac{1}{lp}\right| \leq \frac{p}{p} = \frac{1}{l} \to 0.$$

In fact we have

$$\frac{p}{(l+1)p} = \frac{1}{l+1} \leq \left|\frac{1}{(l-1)p+1} + \cdots + \frac{1}{lp}\right| \leq \frac{1}{l}$$

so the terms of the alternating sequence are decreasing. So this series satisfies the alternating series test, so it is convergent.

Assume now that the (p, q)-Harmonic series converges. We have the subsequence of that sequence of partial sums

$$\begin{aligned} S_{l(p+q)} &= \left(1 + \cdots + \frac{1}{p} - \frac{1}{p+1} - \cdots - \frac{1}{p+q}\right) + \cdots \\ &+ \left(\frac{1}{(l-1)(p+q)+1} + \cdots + \frac{1}{(l-1)(p+q)+p} - \frac{1}{(l-1)(p+q)+p+1} - \cdots - \frac{1}{l(p+q)}\right) \\ &\geq \frac{lp - lq}{lp + lq} = \frac{p-q}{p+q} \end{aligned}$$

We have assumed that the sequence of partial sums is convergent so we must have that any subsequence is convergent so these terms must go to zero which means

$$\lim_{l \to \infty} \frac{p-q}{p+q} = 0$$

which implies that $p = q$.

An alternate proof analyses to two subsequence of partial sums S_{2pn} and S_{2pn+p} and shows that one is monotone increasing and one is montone decreasing for $p = q$ and that for $p \neq q$ we can write the sum of two series (the $p = q$ part and some other part). Since we know already that the $p = q$ part converges, adn by directly analyzing the other part is like the harmonic series so diverges we have a contradiction to the existence of the other part. □

Problem 3. The boundary of a set A, denoted ∂A, is the set of points $x \in \mathbb{R}$ where for every $\epsilon > 0$, $V_\epsilon(x)$ contains a point in A and a point not in A.

(a) Show that A is closed if and only if it contains its boundary.

(b) Show that A is open if and only if it is disjoint from its boundary.

(c) Show that $\partial A = \partial A^c$.

Proof. First note that the boundary of A is the set of points of A which are either limit points of A or isolated points of A. Let $x \in \partial A$. Then every $V_\epsilon x \cap A \neq \emptyset$. So the intersection either contains only x or contains another point of A. These two cases are exactly the definitions of isolated points and limit points respectively.

(a) If A is closed it contains all of its limit points so it contains its boundary. If A contains its boundary then A contains all of its limit points, so it is closed.

(b) Using (c) we see that A^c is closed if and only if it contains its boundary which is if and only if A is disjoint from its boundary.

(c) Points in the boundary of A are points whose ϵ neighborhoods contain a point in A and A^c. And the points in the boundary of A^c are points whose ϵ neighborhoods contain a point of A and a point of A^c. Hence such points satisfy the definition of boundary for both sets.

□

Problem 4. Let $A \subseteq \mathbb{R}$ be a non-empty bounded set. Define $-A = \{-a \mid a \in A\}$. Show that $\sup(-A) = -\inf A$.

Proof. Since A is bounded it has an infimum. Let $s = \inf A$. Since A is non-empty, $-A$ is non-empty. Since A is bounded that means $\exists M > 0$ such that $|a| < M$ for all $a \in A$. Hence $|a| < M$ for all $a \in -A$. Hence $-A$ is bounded. Therefore the supremum of $-A$ exists.

By the definition of infimum $s \leq a$ for all a in A, so we have $-s \geq a$ for all $a \in -A$ Hence $-s$ is an upper bound for $-A$. By the characterization of infimum, for any $\epsilon > 0$ there exists an $a \in A$ such that $a < s + \epsilon$ and for all $a \in A$ $s \leq a$. So we have for any $\epsilon > 0$ there exists an $a \in -A$ such that $a > -s - \epsilon$ and hence by the characterization of supremum, $-s$ is the supremum of $-A$. □

Problem 5. Determine if the following sequences are Cauchy.

(a) The sequence (a_n) for $a_n = \frac{1}{4} + \cdots + \frac{n^2}{4^n}$ for all $n \in \mathbb{N}$.

(b) The sequence (b_n) for $b_n = \frac{1}{2^2} + \cdots + \frac{n}{(n+1)^2}$ for all $n \in \mathbb{N}$.

Proof. (a) (a_n): You can show by induction that $4^n > n^4$ for all $n \geq 5$. Hence

$$|a_{n+k} - a_k| < \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+k)^2}.$$

Consequently,

$$\begin{aligned} |a_{n+k} - a_k| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+k-1)(n+k)} \\ &= \frac{1}{n} - \frac{1}{n-k} < \frac{1}{n}. \end{aligned}$$

So for $n > \frac{1}{\epsilon}$ and any $k \in \mathbb{N}$ we have

$$|a_{n+k} - a_k| < \epsilon.$$

This sequence is Cauchy.

Or you can note that this is the sequence of partial sums of an infinite series and apply the ratio test to see that it is convergent and hence Cauchy.

(b) (b_n): We have

$$\begin{aligned} a_{2n} - a_n &= \frac{2n}{(2n+1)^2} + \cdots + \frac{n+1}{(n+2)^2} \\ &\geq n\frac{2n}{(2n+1)^2} \\ &\geq \frac{2n^2}{(3n)^2} = \frac{2}{9}. \end{aligned}$$

So this is not Cauchy.

or you can note that this is the sequence of partial sums of an infinite series and show that it diverges with the comparison test to the Harmonic series and hence is not Cauchy.

□

Problem 6. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ are finite, then f is uniformly continuous on $\mathbb{R}$.

Proof. Set $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = l$. Then given $\epsilon > 0$ there is $A > 0$ such that $|f(x) - L| < \frac{\epsilon}{2}$ for $x \geq A$ and $|f(x) - l| < \frac{\epsilon}{2}$ for $x \leq -A$. This implies that if $x, y \in [A, \infty)$ or $x, y \in (-\infty, A]$, then $|f(x) - f(y)| < \epsilon$. Since a continuous function on a compact set is continuous we also have that f is uniformly continuous on $[-A, A]$. So f is uniformyl continuous on the union of the these three intervals, which makes f uniformyl continuous on all of $\mathbb{R}$. □

Problem 7. Let (X, d) be a non-empty complete metric space. Suppose $f : X \to X$ is a contraction, i.e. there exists a constant $0 < k < 1$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$, $x \neq y$, and $g : X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point.

Proof. We first show that f has a unique fixed point. We first see that f is continuous. To see this let $\epsilon > 0$ and let $\delta = \epsilon$. Then we have

$$d(f(x), f(y)) \leq kd(x, y) < d(x, y) < \delta = \epsilon.$$

Now we will see that given any $x \in X$ that the sequence $(x_n) = \{x, f(x), f^2(x), \ldots\}$ is Cauchy by induction. We first show that

$$d(x_{n+1}, x_n) \leq k^{n-1} d(x_n, x_{n-1}).$$

The base case is from the definition of contraction, so now note that

$$d(x_{n+2}, x_{n+1}) \leq kd(x_1 n + 1, x_n) \leq k^{n-1} d(x_1, x_2).$$

Now consider $d(x_m, x_n)$ for arbitrary $n \geq m \in \mathbb{N}$. We have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n-1}) + \cdots + d(x_{m+2}, x_{m+1}) \\ &\leq k^{n-1} d(x_1, x_2) + \cdots + k^{m-1} d(x_1, x_2) \\ & k^{m-1}(1 + k + \cdots + k^{n-m-1}) d(x_1, x_2) \\ &< \frac{k^{m-1}}{1-k} d(x_1, x_2). \end{aligned}$$

where we have used the triangle inequality and the sum of a convergent geometric series (since $|k| < 1$). Since $d(x_1, x_2)$ is a constant we can find an N that makes this last quantity as small as we wish. Hence the sequence is Cauchy and hence (x_n) converges to some value y. Note also that $(f(x_n)) \subseteq (x_n)$ we have that $(f(x_n))$ also converges and must converge to the same value y. Hence y is a fixed point of f. We will now show that this is the unique fixed point. Assume that y, z are both fixed points of f. then $d(f(x), f(y)) = d(x, y)$ and $d(f(x), f(y)) \leq kd(x, y) < d(x, y)$ so we must have $d(x, y) = 0$.

Now we will show that y is a fixed point of g. We have $f(y) = y$ and hence $g(f(y)) = g(y)$. Since the functions commute we then have $f(g(y)) = g(y)$ and hence $g(y)$ is a fixed point for f. But since f has a unique fixed point y, we must have $g(y) = y$. □

Problem 8. Suppose that $f : [a, b] \to \mathbb{R}$ and that $f(x) = 0$ for all but finitely many x.

(a) Given $\epsilon > 0$ show that there is a partition P of $[a, b]$ such that $U(f, P) < \epsilon$ and $L(f, P) > -\epsilon$.

(b) Prove that f is Riemann-integrable, and that $\int_a^b f = 0$.

Proof.

(a) Let $\{x_1, \ldots, x_r\}$ be the finitely many real numbers where $f(x) \neq 0$. Let $M = \max_i(|f(x_i|)$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $M\frac{b-a}{N} < \frac{\epsilon}{r}$. Then consider the partition P dividing $[a, b]$ into N equal intervals P_i. We have

$$U(f, P) = \sum_{k=1}^{N} \sup(f(x) \mid x \in P_k) < \sum_{k=1}^{N} \frac{\epsilon}{N} = \epsilon$$
$$L(f, P) = \sum_{k=1}^{N} \inf(f(x) \mid x \in P_k) > \sum_{k=1}^{N} -\frac{\epsilon}{N} = -\epsilon$$

(b) Let $\epsilon > 0$. The function f is clearly bounded, so we need to show that there exists a partition P_ϵ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. From the previous part we can find a partition P such that

$$U(f, P) < \frac{\epsilon}{2}$$
$$L(f, p) > -\frac{\epsilon}{2}.$$

Let $P = P_\epsilon$ and then

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence f is integrable on $[a, b]$. Since we also have that $U(f, P) \to 0$ as $N \to \infty$ we have $U(f) = 0$ and hence $\int_a^b = 0$.

□

Problem 9. Show that the sequence of functions (f_n) with $f_n = x^n(1-x)$ for $n \in \mathbb{N}$ converges uniformly on $[0, 1]$.

Proof. Using the Cauchy Criterion for Uniform Convergence let $\epsilon > 0$. We need to find an $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have

$$|x^m(1-x) - x^n(1-x)| < \epsilon.$$

We have

$$|x^m(1-x) - x^n(1-x)| = (1-x)^{m+1}\left|1 - x^{n-m}\right|.$$

If $x = 0$ ro$x = 1$, this quantity is 0 and clearly less than ϵ. If $x \neq 0, 1$ then we have

$$\left|x^{m+1}\right|\left|1 - x^{n-m}\right| < \left|x^{m+1}\right|$$

Since $x \in (0,1)$ we can find an N such that $x^{m+1} < \epsilon$ for all $m \geq N$ and hence

$$\left|x^{m+1}(1-x) + \cdots x^n(1-x)\right| < \epsilon$$

for all $n > m \geq N$. □

Problem 10. Let R_1 and R_2 be the radii of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ respectively. Show that the radius of convergence R of $\sum_{n=0}^{\infty}(a_n + b_n)x^n$ satisfies $R \geq \min(R_1, R_2)$. Show by example that the inequality can be strict.

Proof. Assume that $x \in (\min(R_1, R_2), \min(R_1, R_2))$. Then we have $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ both converge and hence $\sum_{n=0}^{\infty}(a_n + b_n)x^n$ by the Algebraic Limit Theorem. Since the radius of convergence of a power series is at least as big as any value of convergence, we have that $R \geq \min(R_1, R_2)$.

To show that the inequality can be strict set $a_n = -1, b_n = 1$ for $n = 0, 1, 2, \ldots$. Then $R_1 = R_2 = 1$ and $R = \infty$. □