Physics 48 - February 15, 2008

New Solar-like System Discovered

Review of Linear Algebra

Smaller Version of the Solar System Is Discovered

By DENNIS OVERBYE , NY Times, February 15, 2008

Astronomers said Wednesday that they had found a miniature version of our own solar system 5,000 light-years across the galaxy — the first planetary system that really looks like our own, with outer giant planets and room for smaller inner planets.

In the newly discovered system, a planet about two-thirds of the mass of Jupiter and another about 90 percent of the mass of Saturn are orbiting a reddish star at about half the distances that Jupiter and Saturn circle our own Sun. The star is about half the mass of the Sun.

The new discovery was made by a different technique that favors planets more distant from their star. It is based on a trick of Einsteinian gravity called microlensing. If, in the ceaseless shifting of the stars, two of them should become almost perfectly aligned with Earth, the gravity of the nearer star can bend and magnify the light from the more distant one, causing it to get much brighter for a few days.

If the alignment is perfect, any big planets attending the nearer star will get into the act, adding their own little boosts to the more distant starlight.

That is exactly what started happening on March 28, 2006, when a star 5,000 light-years away in the constellation Scorpius began to pass in front of one 21,000 light-years more distant, causing it to flash.

Vector Space

Note: The concept of a vector space is more general than that of our usual spatial vectors. Space vectors and 4-vectors are special cases. Many other entities (such as our wave functions) will also be members of a vector space.

Vector Space - Any set of vectors $\left(|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \cdots\right)$ and scalars (a, b, c, ...) which are subject to vector addition and scalar multiplication.

Vector Addition

The sum of any two vectors is a vector.

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

-Addition is **Commutative**

$$|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$$

- Addition is **Associative**

$$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$$

- There exists a null vector

$$|\alpha\rangle + |0\rangle = |\alpha\rangle \text{ for all } |\alpha\rangle.$$

- An inverse vector $|-\alpha\rangle$ exists for all $|\alpha\rangle$ such that

$$|\alpha\rangle + |-\alpha\rangle = |0\rangle$$

Scalar Multiplication

The product of a scalar and a vector is a vector $a|\alpha\rangle = |\gamma\rangle$

- Is distributive with respect to vector addition

$$a\left(|\alpha\rangle + |\beta\rangle\right) = a|\alpha\rangle + a|\beta\rangle$$

- and is distributive with respect to scalar addition

$$(a+b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$$

- and is associative with respect to scalar multiplication

$$a\left(b|\alpha\rangle\right) = (ab)|\alpha\rangle$$

-and $0|\alpha\rangle = |0\rangle$ while $1|\alpha\rangle = |\alpha\rangle$

Note that $(1-1)|\alpha\rangle = |0\rangle = |\alpha\rangle - |\alpha\rangle = |\alpha\rangle + |-\alpha\rangle$ and hence

$$|-\alpha\rangle = -1|\alpha\rangle$$

Linear Combinations of Vectors

$$a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \cdots$$

A vector $|\lambda\rangle$ is said to be **linearly independent** of $|\alpha\rangle, |\beta\rangle, |\gamma\rangle$ if it can not be expressed as a linear combination of them.

If all vectors of the space are linearly independent of one another then the set of vectors is said to be **linearly independent.**

A set of vectors is said to **span the space** or be **complete** if every vector can be written as a linear combination of the members of the set.

A set of **linearly independent** vectors that **span the space** is called a **basis set**. The number of members in a basis set is called the **dimension** of the space.

We begin with a finite n dimensional basis set

$$|e_1\rangle, |e_2\rangle, |e_3\rangle, \cdots, |e_n\rangle$$

We can represent any vector $|\alpha\rangle$ as

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + a_3|e_3\rangle + \cdots + a_n|e_n\rangle$$

With the chosen basis set then the vector can be represented as an n-tuple of components

$$|\alpha\rangle \Rightarrow (a_1, a_2, a_3, \cdots, a_n)$$

$$|\alpha\rangle + |\beta\rangle \Rightarrow (a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots, a_n + b_n)$$

and

$$c|\alpha\rangle \Rightarrow (ca_1, ca_2, ca_3, \cdots, ca_n)$$

$$|0\rangle \Rightarrow (0,0,0,0\cdots 0)$$

$$|-\alpha\rangle \Rightarrow (-a_1, -a_2, -a_3, \cdots, -a_n)$$

Inner Products

An inner product of two vectors $|\alpha\rangle$ and $|\beta\rangle$ is defined as a complex number with the properties:

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$$

Note that $\langle\alpha|\alpha\rangle = \langle\alpha|\alpha\rangle^* = \text{real}$

We require that $\langle\alpha|\alpha\rangle \geq 0$ and that $\langle\alpha|\alpha\rangle = 0 \Leftrightarrow |\alpha\rangle = |0\rangle$

And that we have the distributive property

$$\langle\alpha|\big(b|\beta\rangle + c|\gamma\rangle\big) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$$

We define the norm of a vector

$$\|\alpha\| = \sqrt{\langle \alpha | \alpha \rangle}$$

If $\|\alpha\| = 1$ we say that $|\alpha\rangle$ is **normalized**.

If $\langle \alpha | \beta \rangle = 0$ we say that the vectors are **orthogonal**.

A set of vectors that satisfy the equation $\langle \alpha_i | \alpha_j \rangle = \partial_{ij}$ is called an **orthonormal** set. (i.e. it is both orthogonal and normalized)

Inner Product Inequalities

Schwarz inequality

$$\left|\langle\alpha|\beta\rangle\right|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$$

You will prove this one in problem A.5.

Triangle Inequality

$$\left\|\left(|\alpha\rangle+|\beta\rangle\right)\right\| \leq \|\alpha\|+\|\beta\|$$

Linear Transforms

Linear Transforms take a vector in a vector space and "transform" it into another vector in the vector space.

$$\hat{T}|\alpha\rangle = |\alpha'\rangle$$

The transformation must be linear:

$$\hat{T}\big(a|\alpha\rangle + b|\beta\rangle\big) = a\big(\hat{T}|\alpha\rangle\big) + b\big(\hat{T}|\beta\rangle\big)$$

Note: Lorentz Transforms and Rotation Operators are examples of Linear Transforms that you might keep in mind.

If we have an n dimensional basis set for the vector space and we know what the transform does to the basis set

$$\hat{T}\left|e_j\right\rangle = \sum_{i=1}^{n} T_{ij}\left|e_i\right\rangle \text{ for } j = 1,2,\cdots n$$

Then we can figure out what it does to any arbitrary vector. If

$$\left|\alpha\right\rangle = \sum_{j=1}^{n} a_j\left|e_j\right\rangle$$

Then we can write

$$\hat{T}\left|\alpha\right\rangle = \sum_{j=1}^{n} a_j\left(\hat{T}\left|e_j\right\rangle\right) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_j T_{ij}\left|e_i\right\rangle =$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} T_{ij} a_j\right)\left|e_i\right\rangle = \left|\alpha'\right\rangle = \sum_{i=1}^{n} a_i'\left|e_i\right\rangle$$

Thus the components of the vector transform as

$$a_i' = \sum_{j=1}^{n} T_{ij} a_j$$

Thus we can write the operator in matrix form as

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

And the component transform can be written in matrix form as

$$\begin{pmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Or more compactly we can write **a' = T a**

where we have defined the column vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$.

Notice that the sum of two linear transforms is what you would expect

$$(\hat{S} + \hat{T})|\alpha\rangle = \hat{S}|\alpha\rangle + \hat{T}|\alpha\rangle.$$

For matrices this just corresponds to summing each entry in the matrix

$$\mathbf{U} = \mathbf{S} + \mathbf{T} \Leftrightarrow U_{ij} = S_{ij} + T_{ij} \; .$$

When we have the product of two linear transforms it can be reduced to a single linear transform. If $\hat{U} = \hat{S}\hat{T}$ where the product implies first operating with $\hat{T}$ and then operating on the resulting vector with $\hat{S}$. That is

$$|\alpha'\rangle = \hat{T}|\alpha\rangle$$

and $$|\alpha''\rangle = \hat{S}|\alpha'\rangle = \hat{S}(\hat{T}|\alpha\rangle) = \hat{S}\hat{T}|\alpha\rangle$$

For matrices the components of this combined transform can be worked out:

$$a_i'' = \sum_{j=1}^{n} S_{ij} a_j' = \sum_{j=1}^{n} S_{ij} \left(\sum_{k=1}^{n} T_{jk} a_k \right) = \sum_{k=1}^{n} U_{ik} a_k$$

Thus $$U_{ik} = \sum_{j=1}^{n} S_{ij} T_{jk}$$

Which is the standard rule for matrix multiplication where we multiply the corresponding elements of the i’th row of **S** by the k’th column of **T** to obtain the ik’th element of **U**.

$$\begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ U_{21} & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & & \vdots \\ U_{n1} & U_{n2} & \cdots & U_{nn} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

Or again more compactly as **U = S T**.

We define the **transpose** of a matrix, $\tilde{T}$ as the same set of element as the original matrix with the rows and columns interchanged:

$$\tilde{T} = \begin{pmatrix} T_{11} & T_{21} & \cdots & T_{n1} \\ T_{12} & T_{22} & \cdots & T_{n2} \\ \vdots & \vdots & & \vdots \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix}$$

We note that the transpose of a column matrix like **a** is a row matrix i.e.

$$\tilde{a} = (a_1 \quad a_2 \quad \cdots \quad a_n)$$

Hermitian Matrices

We define the Hermitian conjugate to be the transpose of the conjugate matrix.

$$T^t = \tilde{T}^*$$

Note: I am using *t* in place of the dagger.

A matrix is Hermitian if it is equal to its Hermitian conjugate. Only a square matrix can be Hermitian.

Hermitian: $T^t = T$

and Skew Hermitian: $T^t = -T$

With these definitions the inner product with respect to a particular basis set can be written as

$$\langle \alpha | \beta \rangle = \mathbf{a}^{\mathrm{t}}\mathbf{b}$$

i.e. $$\left(a_1^*, a_2^*, \cdots a_n^*\right)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Notice that matrix multiplication is not commutative. We define the commutator as:

$$[\mathbf{S}, \mathbf{T}] = \mathbf{ST} - \mathbf{TS}$$

Identity and Inverse Matrix

The unit matrix is $\mathbf{I} = \begin{pmatrix} 1 & 0 \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & 1 \end{pmatrix}$

i.e. $I_{ij} = \delta_{ij}$

The inverse matrix is defined such that $\mathbf{T}^{-1}\,\mathbf{T} = \mathbf{T}\,\mathbf{T}^{-1} = \mathbf{I}$

It can be shown that a matrix has an inverse iff the $\det(\mathbf{T}) \neq 0$.

Recall that the determinant of a matrix is

$$\det(\mathbf{T}) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

and is calculated in the usual manner.

You will get to practice your matrix manipulations in A.8, A.9, and A.14.

Eigenvectors and eigenvalues

If a linear transform leaves a particular non-null vector $|\alpha\rangle$ unaltered (multiplied only by a constant, complex coefficient)

$$\mathbf{T}|\alpha\rangle = \lambda|\alpha\rangle$$

we say the $|\alpha\rangle$ is an *eigenvector* of the transform **T** and that the complex number, λ is called an *eigenvalue*.

e.g. – for rotations about the $\hat{x}$ axis a vector along $\hat{x}$ is unchanged. Therefore it is an eigenvector of this rotation operator.

In a complex vector space, every linear transform has such vectors. For a particular basis set we may write the matrix equation

$$\mathbf{Ta} = \lambda\mathbf{a}$$

$$(\mathbf{T} - \lambda\mathbf{I})\,\mathbf{a} = \mathbf{0}$$

The characteristic equation

If $(\mathbf{T} - \lambda\mathbf{I})$ has an inverse, then $(\mathbf{T} - \lambda\mathbf{I})^{-1} (\mathbf{T} - \lambda\mathbf{I})\, \mathbf{a} = (\mathbf{T} - \lambda\mathbf{I})^{-1}\, \mathbf{0}$

$$\mathbf{I}\,\mathbf{a} = \mathbf{0}$$

Which implies that $\mathbf{a} = \mathbf{0}$. (*not too interesting*).

If $(\mathbf{T} - \lambda\mathbf{I})$ does not have an inverse then

$$\det(\mathbf{T} - \lambda\mathbf{I}) = 0. \quad (\text{assuming } \mathbf{a} \neq \mathbf{0}).$$

i.e. $$\begin{pmatrix} (T_{11}-\lambda) & T_{12} & T_{13} & \cdots & T_{1n} \\ T_{21} & (T_{22}-\lambda) & T_{23} & \cdots & T_{2n} \\ \vdots & & \ddots & & \vdots \\ T_{n1} & T_{n2} & & \cdots & (T_{nn}-\lambda) \end{pmatrix} = 0$$

This is a polynomial of order n in λ, called the ***characteristic equation***.

Determination of the eigenvalues and eigenvectors

The characteristic equation looks something like

$$C_n\lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda^1 + C_0 = 0$$

There are between 1 and n eigenvalues, λ that will solve the equation.

For each eigenvalue, one can then return to the equation $\mathbf{T}|\alpha\rangle = \lambda|\alpha\rangle$ to determine the eigenvector(s) that correspond to the particular eigenvalue.

You will practice this in problem A.26.