Physics 48, February 18, 2008

- Eigenvalues and eigenvectors
- Hermitian transformations
- Function spaces
- Hilbert Space
- Hermitian Operators

Review

Last class we discussed the general characteristics of vector spaces and defined the inner product.

We found that for vector spaces with finite dimension, n, we could represent the state vectors as n dimensional vectors and transformations by n x n matrices.

We explored a number of characteristics of this representation and defined the Hermitian conjugate of a matrix to be the conjugate transpose.

The inner product in an n-dimensional space was found to be

$$\langle \alpha | \beta \rangle = \mathbf{a}^{t}\mathbf{b}$$

Eigenvectors and eigenvalues

If a linear transform leaves a particular non-null vector $|\alpha\rangle$ unaltered (multiplied only by a constant, complex coefficient)

$$\mathbf{T}|\alpha\rangle = \lambda|\alpha\rangle$$

we say the $|\alpha\rangle$ is an *eigenvector* of the transform **T** and that the complex number, λ is called an *eigenvalue*.

e.g. – for rotations about the $\hat{x}$ axis a vector along $\hat{x}$ is unchanged. Therefore it is an eigenvector of this rotation operator.

In a complex vector space, every linear transform has such vectors. For a particular basis set we may write the matrix equation

$$\mathbf{Ta} = \lambda\mathbf{a}$$

$$(\mathbf{T} - \lambda\mathbf{I})\,\mathbf{a} = \mathbf{0}$$

The characteristic equation

If $(\mathbf{T} - \lambda\mathbf{I})$ has an inverse, then $(\mathbf{T} - \lambda\mathbf{I})^{-1} (\mathbf{T} - \lambda\mathbf{I})\,\mathbf{a} = (\mathbf{T} - \lambda\mathbf{I})^{-1}\,\mathbf{0}$

$$\mathbf{I}\,\mathbf{a} = \mathbf{0}$$

Which implies that $\mathbf{a} = \mathbf{0}$. (*not too interesting*).

If $(\mathbf{T} - \lambda\mathbf{I})$ does not have an inverse then

$\det(\mathbf{T} - \lambda\mathbf{I}) = 0$. (assuming $\mathbf{a} \neq \mathbf{0}$).

i.e. $$\begin{pmatrix} (T_{11} - \lambda) & T_{12} & T_{13} & \cdots & T_{1n} \\ T_{21} & (T_{22} - \lambda) & T_{23} & \cdots & T_{2n} \\ \vdots & & \ddots & & \vdots \\ T_{n1} & T_{n2} & & \cdots & (T_{nn} - \lambda) \end{pmatrix} = 0$$

This is a polynomial of order n in λ, called the ***characteristic equation***.

Determination of the eigenvalues and eigenvectors

The characteristic equation looks something like

$$C_n \lambda^n + C_{n-1} \lambda^{n-1} + \cdots + C_1 \lambda^1 + C_0 = 0$$

There are between 1 and n eigenvalues, λ that will solve the equation.

For each eigenvalue, one can then return to the equation $\mathbf{T}|\alpha\rangle = \lambda|\alpha\rangle$ to determine the eigenvector(s) that correspond to the particular eigenvalue.

You will practice this in problem A.26.

Hermitian Transformations

For a matrix, we defined the Hermitian conjugate of a matrix to be the conjugate transpose

$$T^t = \tilde{T}^*$$

(reminder: I am using t for the dagger)

More generally, for any linear operator we define the Hermitian conjugate operator by the relation

$$\boxed{\langle T^t\alpha | \beta\rangle \equiv \langle \alpha | T\beta\rangle}$$

Note consistency with matrix definition of Hermitian operator

$$(\mathbf{T}^{\dagger}\mathbf{a})^{\dagger}\mathbf{b} = \left(\begin{pmatrix} T_{11}{}^{*} & T_{21}{}^{*} & \cdots & T_{n1}{}^{*} \\ T_{12}{}^{*} & T_{22}{}^{*} & \cdots & T_{n2}{}^{*} \\ \vdots & \vdots & & \vdots \\ T_{1n}{}^{*} & T_{2n}{}^{*} & \cdots & T_{nn}{}^{*} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \right)^{\dagger} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} =$$

Recall for matrices $(ST)^{\dagger} = T^{\dagger}S^{\dagger}$

$$(a_1{}^{*} \quad a_2{}^{*} \quad . \quad . \quad a_n{}^{*}) \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \mathbf{a}^{\dagger}\mathbf{T}\mathbf{b}$$

$$\langle \alpha | T\beta \rangle = \mathbf{a}^{\dagger}\mathbf{T}\mathbf{b} = (\mathbf{T}^{\dagger}\mathbf{a})^{\dagger}\mathbf{b} = \langle T^{\dagger}\alpha | \beta \rangle$$

Properties of constants in the inner product

Note $$\langle \alpha | c\beta \rangle = c\langle \alpha | \beta \rangle$$

(from linearity property of the inner product)

and $$\langle c\alpha | \beta \rangle = \langle \beta | c\alpha \rangle^* .$$

(one of the defining characteristics of the inner product).

$$\langle c\alpha | \beta \rangle = \left(c\langle \beta | \alpha \rangle\right)^* = c^* \langle \beta | \alpha \rangle^*$$

(linearity again and the complex conjugate of a product).

$$\langle c\alpha | \beta \rangle = c^* \langle \alpha | \beta \rangle$$

Properties of Hermitian Transformations

1. The eigenvalues of a Hermitian Transform are real.

Proof: Let λ be an eigenvalue of T s.t. $T|\alpha\rangle = \lambda|\alpha\rangle$ with $|\alpha\rangle \neq |0\rangle$.

Then $\langle\alpha|T\alpha\rangle = \langle\alpha|\lambda\alpha\rangle = \lambda\langle\alpha|\alpha\rangle$.

If T is Hermitian then $\langle\alpha|T\alpha\rangle = \langle T^t\alpha|\alpha\rangle = \langle T\alpha|\alpha\rangle = \lambda^*\langle\alpha|\alpha\rangle$

So if $\langle\alpha|\alpha\rangle \neq 0 \Rightarrow \lambda = \lambda^* \Rightarrow \lambda$ is real.

2. The eigenvectors of a Hermitian transformation belonging to distinct eigenvalues are orthogonal.

Proof: Let $T|\alpha\rangle = \lambda_A|\alpha\rangle$ and let $T|\beta\rangle = \lambda_B|\beta\rangle$

$$\langle\alpha|T\beta\rangle = \langle\alpha|\lambda_B\beta\rangle = \lambda_B\langle\alpha|\beta\rangle$$

$$\langle\alpha|T\beta\rangle = \langle T^t\alpha|\beta\rangle = \langle T\alpha|\beta\rangle \quad \text{(Hermitian)}$$

$$\langle\alpha|T\beta\rangle = \langle\lambda_A\alpha|\beta\rangle = \lambda_A{}^*\langle\alpha|\beta\rangle$$

$$\langle\alpha|T\beta\rangle = \lambda_A\langle\alpha|\beta\rangle \quad \text{(eigenvalues are real)}$$

So $\lambda_B\langle\alpha|\beta\rangle = \lambda_A\langle\alpha|\beta\rangle$

But $\lambda_B \neq \lambda_A$, so $\langle\alpha|\beta\rangle = 0$, i.e. they are orthogonal.

3. The eigenvectors of a Hermitian transformation span the space.

If all n roots of the characteristic equation are distinct, then 2 implies that these constitute n mutually orthogonal eigenvectors. This implies that they must span the space.

Also true when the roots are not distinct (i.e. when we have degenerate eigenvalues). We will not prove this.

Function Spaces

Function spaces are a type of vector space.

Vectors correspond to complex functions of x.

Inner products correspond to integrals.

Linear transforms correspond to operators.

Do functions satisfy the requirements of a “vector space”?

Is the sum of two functions a function? (Is it in the space?)

Is cf(x) a function? (is it in the space?)

Is there a null? (f(x) = 0).

We will encounter many classes of functions, but whatever class we have must satisfy the above conditions.

Inner product in a function space

We define the inner product for a function space to be

$$\langle f \mid g \rangle \equiv \int f^*(x) g(x) dx$$

Does this satisfy the three conditions of the inner product?

i) Is $\langle \beta \mid \alpha \rangle = \langle \alpha \mid \beta \rangle^*$?

ii) Is $\langle \alpha \mid \alpha \rangle \geq 0$? Does it only = 0 when $|\alpha\rangle$ = **0**?

iii) Is $\langle \alpha \mid \left(b \mid \beta \rangle + c \mid \gamma \rangle \right) = b\langle \alpha \mid \beta \rangle + c\langle \alpha \mid \gamma \rangle$?

Hilbert Space

In Physics, we restrict ourselves to square-integrable functions.

i.e. Functions such that $\int |f(x)|^2 dx < \infty$. This is Hilbert Space (a subset of all function spaces).

Quantum Mechanical wave functions live in Hilbert Space.

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Note: If our functions are square integrable, then the Schwartz inequality suggests that the inner product of two functions is also finite.

$$|\langle f | g \rangle|^2 \le \langle f | f \rangle \langle g | g \rangle .$$

A complete orthonormal set of functions

A set of functions $\{f_n\}$ is said to be orthonormal if $\langle f_m | f_n \rangle = \delta_{mn}$.

The set is complete if any function *f(x)* can be expressed as a linear combination of the the set.

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

The generalization of Fourier's trick

Notice what happens when we take the inner product of any function with one of the orthonormal basis functions.

$$\langle f_m(x) | f(x) \rangle = \left\langle f_m(x) \middle| \sum_{n=1}^{\infty} c_n f_n(x) \right\rangle = \sum_{n=1}^{\infty} c_n \langle f_m(x) | f_n(x) \rangle$$

$$\langle f_m(x) | f(x) \rangle = \sum_{n=1}^{\infty} c_n \delta_{mn}$$

$$\boxed{c_m = \langle f_m(x) | f(x) \rangle}$$

So the projections on to the basis set are again given by Fourier's trick.

Correspondences Between Vectors in $\mathcal{E}_3$ and Vectors in $\mathcal{H}$

Item	$\mathcal{E}_3$	$\mathcal{H}$
ector	Directed line segment, $\mathbf{v}$	Complex function, $\psi(x)$
calar	Real number, r	Complex number, c
inear combination	$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$	$\psi(x) = c_1\psi_1(x) + c_2\psi_2(x)$
nner product[1]	$\mathbf{v}_1 \cdot \mathbf{v}_2 \equiv \lvert\mathbf{v}_1\rvert\,\lvert\mathbf{v}_2\rvert \cos\theta_{12}$	$(\psi_1,\psi_2) \equiv \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)dx$
orm[2]	$\mathbf{v}\cdot\mathbf{v} = \lvert\mathbf{v}\rvert^2$	$(\psi,\psi) = \int_{-\infty}^{\infty} \lvert\psi(x)\rvert^2 dx$
roperties of the inner product	$\mathbf{v}_2\cdot\mathbf{v}_1 = \mathbf{v}_1\cdot\mathbf{v}_2$ $r_1\mathbf{v}_1\cdot r_2\mathbf{v}_2 = r_1 r_2 \mathbf{v}_1\cdot\mathbf{v}_2$ $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_3+\mathbf{v}_4)$ $= \mathbf{v}_1\cdot\mathbf{v}_3 + \mathbf{v}_1\cdot\mathbf{v}_4 + \mathbf{v}_2\cdot\mathbf{v}_3 + \mathbf{v}_2\cdot\mathbf{v}_4$ $\lvert\mathbf{v}_1\cdot\mathbf{v}_2\rvert \leqq \sqrt{\mathbf{v}_1\cdot\mathbf{v}_1}\sqrt{\mathbf{v}_2\cdot\mathbf{v}_2}$	$(\psi_2,\psi_1) = (\psi_1,\psi_2)^*$ $(c_1\psi_1, c_2\psi_2) = c_1^* c_2(\psi_1,\psi_2)$ $(\psi_1+\psi_2, \psi_3+\psi_4)$ $= (\psi_1,\psi_3) + (\psi_1,\psi_4) + (\psi_2,\psi_3) + (\psi_2,\psi_4)$ $\lvert(\psi_1,\psi_2)\rvert \leqq \sqrt{(\psi_1,\psi_1)}\sqrt{(\psi_2,\psi_2)}$
rthogonal vectors	$\mathbf{v}_1\cdot\mathbf{v}_2 = 0$	$(\psi_1,\psi_2) = 0$
rthonormal basis set[3]	$\{\mathbf{e}_i\}$, with $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$ and also, for any $\mathbf{v}$ in $\mathcal{E}_3$, $\mathbf{v} = \sum_{i=1}^{3}(\mathbf{e}_i\cdot\mathbf{v})\mathbf{e}_i$	$\{\epsilon_i(x)\}$, with $(\epsilon_i,\epsilon_j) = \delta_{ij}$ and also, for any $\psi(x)$ in $\mathcal{H}$ $\psi(x) = \sum_{i=1}^{\infty}(\epsilon_i,\psi)\epsilon_i(x)$

[1] The inner product of two vectors is a *scalar*.
[2] The norm of a vector is a *nonnegative real number*.
[3] The scalars $\{\mathbf{e}_i\cdot\mathbf{v}\}$ and $\{(\epsilon_i,\psi)\}$ are called the *expansion coefficients* or *components* of the vectors $\mathbf{v}$ and $\psi(x)$ relativ
o the respective bases.