Math 28 2009: Final Exam

Instructions:

Problem 1. Let $(a_n) \subset \mathbb{R}$ be a sequence. Let A be the set of limit points of subsequences of (a_n). In other words, $a \in A$ if and only if there exists a subsequence (a_{n_k}) such that $(a_{n_k}) \to a$. Show that A is closed.

Problem 2. Determine the cardinality of the set $\{(x, y) \mid x, y \in \mathbb{Q}, x^2 + y^2 = 1\}$. (Hint: consider lines through $(0, 1)$.)

Problem 3. Recall that we say $f : \mathbb{R} \to \mathbb{R}$ is *periodic* with period T if $f(x + T) = f(x)$ for all x.

(a) Show that if f is continuous and periodic then it attains its supremum and infimum.

(b) Prove that any function that is continuous and periodic must be uniformly continuous.

Problem 4. Let $A, B \subset \mathbb{R}$ be nonempty disjoint compact sets. Show that $A \cup B$ is *not* connected.

Problem 5. Let (X, d) be a metric space and let $f_n : X \to X$ be uniformly continuous for all $n \in \mathbb{N}$. Show that if (f_n) converges uniformly on X, then the limit function is also uniformly continuous on X.

Problem 6. Let $f : (a, b) \to \mathbb{R}$ be continuous. Prove that given $x_1, \ldots, x_n$ in (a, b) that there exists an $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{1}{n} \left(f(x_1) + \cdots + f(x_n) \right).$$

Problem 7. Let $f : (a, b) \to \mathbb{R}$. Given $c \in (a, b)$, show that f is differentiable at c if and only if there exists a constant M so that

$$f(x) = f(c) + M(x - c) + r(x)$$

where $r(x)$ satisfies

$$\lim_{x \to c} \frac{r(x)}{x - c} = 0.$$

Problem 8. Suppose that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A and that $g : A \to R$ is bounded.

(a) Prove that the series $\sum_{n=1}^{\infty} g(x) f_n(x)$ converges uniformly on A.

(b) Show by example that the boundedness of g is necessary for part (a).