Solutions to the Analysis problems on the Comprehensive Examination of January 27, 2012

1. [4 points] State the Axiom of Completeness (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

   Solution: Every nonempty set of real numbers that is bounded above has a least upper bound.

2. (a) [6 points] The standard triangle inquality states that $|x+y| \leq |x|+|y|$ for $x, y \in \mathbb{R}$. Assuming this result, give a careful proof that if $x_1, \dots, x_n \in \mathbb{R}$, $n \geq 2$, then $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$.

   Solution: We can prove this by induction on n.

   Base Case: When $n = 2$, we already know $|x + y| \leq |x| + |y|$. Thus the inequality is true when $n = 2$.

   Now suppose the inequality holds for some n; then we have $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$. Now we can show that the same is true for $n + 1$:

   $$|x_1 + \cdots + x_{n+1}| \leq |x_1 + \cdots + x_n| + |x_{n+1}| \leq |x_1| + \cdots + |x_n| + |x_{n+1}|.$$

   Therefore the inquality is true for $n+1$ as well. Combining the base case and the inductive step we can conclude that if $x_1, \dots, x_n \in \mathbb{R}$, $n \geq 2$, then $|x_1+\cdots+x_n| \leq |x_1| + \cdots + |x_n|$.

   (b) [6 points] Recall that a function $f : S \to \mathbb{R}$ is bounded if there is $M \in \mathbb{R}$ with $|f(x)| \leq M$ for all $x \in S$. Now suppose we have bounded functions $f_1, \dots, f_n : S \to \mathbb{R}$, $n \geq 2$, and define $f_1 + \cdots + f_n : S \to \mathbb{R}$ by $(f_1 + \cdots + f_n)(x) = f_1(x) + \cdots + f_n(x)$ for $x \in S$. Prove that $f_1 + \cdots + f_n$ is bounded.

   Solution: Because $f_1, \dots, f_n$ are bounded, there exist $M_1, \dots, M_n \in \mathbb{R}$ such that $|f_k(x)| \leq M_k$ for all $x \in S$ and $1 \leq k \leq n$. Let $M = M_1 + \cdots + M_n$. By the theorem we just proved in (a) we have:

   $$\begin{aligned} |(f_1 + \cdots + f_n)(x)| &= |f_1(x) + \cdots + f_n(x)| \\ &\leq |f_1(x)| + \cdots + |f_n(x)| \leq M_1 + \cdots + M_n = M \end{aligned}$$

   for all $x \in S$. Hence the function $f_1 + \cdots + f_n$ is bounded by M.

3. Consider the sequence of functions defined by $f_n(x) = 2 + (1 + 1/n)x$ for $n \geq 1$. This sequence converges pointwise to $f(x) = 2 + x$.

   (a) [7 points] Prove that the sequence converges uniformly to f on $[0, 10]$.

Solution: To say that the sequence converges uniformly to f on $[0, 10]$ means that for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ is true for any $n \geq N$, $x \in [0, 10]$.

The key observation is that $|f_n(x) - f(x)| = |\frac{x}{n}| \leq \frac{10}{n}$ for $x \in [0, 10]$. This makes it easy to find an N that satisfies our conditions given $\epsilon > 0$. Here is the proof.

Given any $\epsilon > 0$, we can choose $N \in \mathbb{N}$ with $N > \frac{10}{\epsilon}$. It follows that for $n \geq N$ and $x \in [0, 10]$,

$$|f_n(x) - f(x)| = |2 + (1 + 1/n)x - (2 + x)| = \left|\frac{x}{n}\right| \leq \frac{10}{n} \leq \frac{10}{N} < \frac{10}{\frac{10}{\epsilon}} = \epsilon.$$

Therefore, the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on [0,10]

(b) [7 points] Prove that the sequence does not converge uniformly on $[0, \infty)$.

Solution: We will prove that the negation of uniform convergence is true. Therefore our goal is to show that there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$, we can find some $x \in [0, \infty)$ satisfying $|f_N(x) - f(x)| > \epsilon$. Below we give the proof.

Pick $\epsilon = 1$. For any $N \in \mathbb{N}$, let $x = N + 1$. Then we have

$$|f_N(x) - f(x)| = \left|2 + (1 + 1/N)x - (2 + x)\right| = \left|\frac{x}{N}\right| = \left|\frac{N+1}{N}\right| > 1.$$

This shows that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0, \infty)$.

4. [10 points] Suppose that we have continuous functions $f, g : \mathbb{R} \to \mathbb{R}$. Prove that the composition $f \circ g : \mathbb{R} \to \mathbb{R}$ is also continuous.

Solution: First note that because $g : \mathbb{R} \to \mathbb{R}$, we know that $g(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. Therefore $(f \circ g)(x) = f(g(x)) \in \mathbb{R}$, so $f \circ g$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

To prove that $f \circ g : \mathbb{R} \to \mathbb{R}$ is continuous, we give an ϵ-δ proof that it is continuous at every $c \in \mathbb{R}$. So fix $c \in \mathbb{R}$.

Suppose $\epsilon > 0$. Since $f : \mathbb{R} \to \mathbb{R}$ is continuous, it is continuous at $g(c)$. Thus, we can find a $\delta_1 > 0$ such that

$$|x - g(c)| < \delta_1 \quad \Rightarrow \quad |f(x) - f(g(c))| < \epsilon.$$

Because $g : \mathbb{R} \to \mathbb{R}$ is also continuous, it is continuous at c. Using the above δ_1, we can find a $\delta_2 > 0$ such that

$$|x - c| < \delta_2 \quad \Rightarrow \quad |g(x) - g(c)| < \delta_1.$$

Combining the above implications, we see that

$$|x - c| < \delta_2 \quad \Rightarrow \quad |g(x) - g(c)| < \delta_1 \quad \Rightarrow \quad |f(g(x)) - f(g(c))| < \epsilon.$$

Therefore $f(g(x)) = (f \circ g)(x)$ is continuous at c. Since this is true for all $c \in \mathbb{R}$, we conclude that $f \circ g : \mathbb{R} \to \mathbb{R}$ is continuous.