Solutions to the Calculus and Linear Algebra problems on the Comprehensive Examination of February 1, 2008

Solutions to Problems 1-5 and 10 are omitted since they involve topics no longer covered on the Comprehensive Examination.

6. Evaluate the following integrals:

   (a) $\displaystyle\int_0^1 \int_{2x}^2 e^{(y^2)}\, dy\, dx.$

   Solution: We need to switch the order of integration. The region of integration lies between $y = 2x$ and $y = 2$ for $0 \le x \le 1$. This is the same as the region between $x = 0$ and $x = y/2$ for $0 \le y \le 2$. Thus

$$\begin{aligned}\int_0^1 \int_{2x}^2 e^{(y^2)}\, dy\, dx &= \int_0^2 \int_0^{y/2} e^{(y^2)}\, dx\, dy \\ &= \int_0^2 xe^{y^2}\Big|_{x=0}^{y/2} dy = \frac{1}{2}\int_0^2 ye^{y^2}\, dy \\ &= \frac{1}{4}e^{y^2}\Big|_0^2 = \frac{1}{4}(e^4 - 1)\end{aligned}$$

   where we used the substitution $u = y^2$, $du = 2ydy$.

   (b) $\int_C (e^x + y^3)dx + (6y^2x + x^3)dy$, where C is the circle $x^2+y^2 = 1$, oriented counterclockwise.

   Solution: Apply Greens's Theorem, letting D be the domain enclosed by C:

$$\int_C (e^x + y^3)dx + (6y^2x + x^3)dy = \iint_D (6y^2 + 3x^2 - 3y^2)\, dA = 3\iint_D (y^2 + x^2)\, dA$$

   Now integrate on D using polar coordinates, easy enough since D is just a circle of radius 1 and the integrand turns into r^2:

$$3\int_0^{2\pi} \int_0^1 r^2 \cdot r\, dr\, d\theta = 3(2\pi)\left(\frac{r^4}{4}\Big|_{r=0}^1\right) = \frac{3\pi}{2}$$

7. Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

   Solution: We'll be working with spherical coordinates here. The inside-the-sphere constraint is pretty obvious: it just means that $\rho \le 2$. The cone constraint is a bound on ϕ. To find the specific bound, it can be helpful to just consider the xz-plane.

Above the cone on that plane means that $z \geq x$, which converted to spherical means $\tan(\phi) \leq 1$, or $\phi \leq \pi/4$. Any θ satisfies the constraints, so integrating:

$$\begin{aligned} V &= \iiint_V dV \\ &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \\ &= 2\pi \Big(\frac{\rho^3}{3} \Big|_0^2 \Big) \Big(-\cos(\phi) \Big|_0^{\pi/4} \Big) \\ &= \frac{16\pi}{3} \left(1 - \frac{\sqrt{2}}{2} \right) \end{aligned}$$

8. Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Find $f_x(0,0)$ and $f_y(0,0)$.

Solution: Use the definition of partial derivative:

$$f_x(0,0) = \lim_{h\to 0} \frac{\frac{h^3}{h^2+0^2}}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

$$f_y(0,0) = \lim_{h\to 0} \frac{\frac{0^3}{0^2+h^2}}{h} = \lim_{h\to 0} \frac{0}{h^3} = 0$$

(b) Use the definition of the directional derivative to find the directional derivative of f at $(0,0)$ in the direction $\vec{u} = (\sqrt(2)/2, \sqrt(2)/2)$.

Solution: From the definition of directional derivative,

$$D_{\vec{u}} = \lim_{h\to 0} \frac{\frac{(h\sqrt{2}/2)^3}{(h\sqrt{2}/2)^2+(h\sqrt{2}/2)^2}}{h} = \lim_{h\to 0} \frac{h^3}{h^3} \frac{(\sqrt{2}/2)^3}{2(\sqrt{2}/2)^2} = \frac{\sqrt{2}}{4}$$

(c) Is f differentiable at (0,0)? Justify your answer.

Solution: Will not need to prove differentiability in the new comps.

9. Find the maximum value of the function $f(x,y) = x^2 y$ on the ellipse $x^2 + 2y^2 = 24$.

Solution: We use Lagrange multipliers here to find the max of $f(x,y) = x^2 y$ with the constraint $g(x,y) = x^2 + 2y^2 = 24$. To find the points to test, we solve the system

$\nabla f(x, y) = \lambda \nabla g(x, y)$ (so $2xy = \lambda(2x)$ and $x^2 = \lambda(4y)$) and $g(x, y) = 24$. We'll break up the system into two cases here. If $x = 0$, then clearly $y = \pm\sqrt{12}$. If $x \neq 0$, then from the first equation $y = \lambda$ so substituting into the second equation, $x^2 = 4y^2$. We now substitute in for x in the last equation: $6y^2 = 24$, which gives us $y = \pm 2$ and $x = \pm 4$ for four more critical points $(\pm 4, \pm 2)$. Now it is just a matter of plugging each of these 6 points into f, and we find that the maximum occurs at $(\pm 4, 2)$ for a value of 32.

11. Let

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 0 & -5 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Find all eigenvalues of A.

Solution: We want to find all possible $\lambda \in \mathbb{R}$ such that $\exists \vec{v} \neq \vec{0} \in \mathbb{R}^3$ such that $A\vec{v} = \lambda\vec{v}$, which means that the system $(A - \lambda I)\vec{v} = \vec{0}$ has a nontrivial solution. Thus, $A - \lambda I$ is non-invertible, or $\det(A - \lambda I) = 0$. We use the recursive formula for the determinant to simplify the calculation:

$$\begin{aligned} \det(A - \lambda I) &= -0 \det \begin{pmatrix} 3 & -5 \\ 1 & -\lambda \end{pmatrix} + (-\lambda) \det \begin{pmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{pmatrix} - 0 \det \begin{pmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= -\lambda(\lambda^2 - 2\lambda + 1) = -\lambda(\lambda - 1)^2 \end{aligned}$$

Thus the eigenvalues of A are 0 and 1.

(b) Find an invertible matrix P such that PAP^{-1} is a diagonal matrix, or show that there is no such matrix P.

Solution: The eigenvalue 1 has algebraic multiplicity 2, so let's start the diagonalization process there: if we don't get an eigenspace of dimension 2, then we know A is not diagonalizable so we don't have to bother with the eigenvalue of 0. Without further ado, we find the nullspace of $A - I$:

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & -1 & -5 \\ 1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

There is only one free variable, which means that the nullspace of $(A - I)$ is only 1. Thus, the sum of the dimensions of the eigenspaces of A is 2 and not 3, so A is not diagonalizable (that is, there is no P such that PAP^{-1} is diagonal).

12. Suppose that u, v, and w are distinct vectors in a vector space V, and $\{u, v, w\}$ is linearly independent. Prove that $\{u + 2v, v + 2w, u + 2w\}$ is linearly independent.

Solution: Given α_1,α_2, and α_3 in $\mathbb{R}$ such that $\alpha_1(u+2v)+\alpha_2(v+2w)+\alpha_3(u+2w) = 0$, we can rearrange the terms to find that $(\alpha_1 + \alpha_3)u + (2\alpha_1 + \alpha_2)v + (2\alpha_2 + 2\alpha_3) = 0$.

Since $\{u, v, w\}$ is linearly independent, $\alpha_1 + \alpha_3 = 0$, $2\alpha_1 + \alpha_2 = 0$, and $2\alpha_2 + 2\alpha_3 = 0$. We now find all possible solutions to that system by considering the nullspace of the matrix corresponding to the system:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the matrix has a trivial nullspace! Thus $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so $\{u+2v, v+2w, u+2w\}$ is linearly independent, as desired. QED

13. Suppose V, W, and Z are finite-dimensional vector spaces, $T : V \to W$ and $U : W \to Z$ are linear transformations, and T is onto.

(a) Prove that $\text{range}(UT) = \text{range}(U)$.

Solution: $\subseteq$:
Given $z \in \text{range}(UT)$, by definition of range $\exists v \in V$ such that $U \circ T(v) = z$. But then $U(T(v)) = z$ so $z \in \text{range}(U)$.✓
$\supseteq$:
Given $z \in \text{range}(U)$, by definition of range $\exists w \in W$ such that $U(w) = z$. But since T is onto, $\exists v \in V$ such that $T(v) = w$. Thus, $U \circ T(v) = z$ so $z \in \text{range}(UT)$.✓
Thus $\text{range}(UT) = \text{range}(U)$ as desired. QED

(b) Prove that $\text{nullity}(UT) = \text{nullity}(U) + \text{nullity}(T)$.

Solution: We use the rank-nullity theorem; there are a lot of functions and vector spaces flying around, so we start by defining some variables. Let $a = \text{nullity}(U)$, $b = \text{nullity}(T)$, $c = \text{nullity}(UT)$, $\alpha = \dim(V)$, $\beta = \dim(W)$, $q = \text{rank}(U)$, $r = \text{rank}(T)$, and $s = \text{rank}(UT)$. Since T is onto, $r = \beta$ and since we already showed that $\text{range}(UT) = \text{range}(U)$, $s = q$. Applying the rank-nullity theorem to U, T, and UT respectively and using those two substitutions, we find:

$$\beta = q + a$$

$$\alpha = \beta + b$$

$$\alpha = q + c$$

Solving for c and substituting, we find:

$$c = \alpha - q = (\beta + b) - q = (q + a) + b - q = a + b$$

as desired. QED