Economics 58

Proof of Envelope Theorem for constrained optimization problems (from Varian)

Consider a parameterized maximization problem of the form

$$M(a) = \max_{x_1, x_2} g(x_1, x_2, a)$$
$$\text{such that } h(x_1, x_2, a) = 0.$$

The Lagrangian for this problem is

$$\mathcal{L} = g(x_1, x_2, a) - \lambda h(x_1, x_2, a),$$

and the first-order conditions are

$$\begin{aligned} \frac{\partial g}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} &= 0 \\ \frac{\partial g}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} &= 0 \\ h(x_1, x_2, a) &= 0. \end{aligned} \tag{27.4}$$

These conditions determine the optimal choice functions $(x_1(a), x_2(a))$, which in turn determine the maximum value function

$$M(a) \equiv g(x_1(a), x_2(a), a). \tag{27.5}$$

The **envelope theorem** gives us a formula for the derivative of the value function with respect to a parameter in the maximization problem. Specifically, the formula is

$$\begin{aligned} \frac{dM(a)}{da} &= \left.\frac{\partial \mathcal{L}(\mathbf{x}, a)}{\partial a}\right|_{\mathbf{x}=\mathbf{x}(a)} \\ &= \left.\frac{\partial g(x_1, x_2, a)}{\partial a}\right|_{\mathbf{x}=\mathbf{x}(a)} - \lambda \left.\frac{\partial h(x_1, x_2, a)}{\partial a}\right|_{\mathbf{x}=\mathbf{x}(a)}. \end{aligned}$$

As before, the interpretation of the partial derivatives needs special care: they are the derivatives of g and h with respect to a *holding x_1 and x_2 fixed at their optimal values.*

The proof of the envelope theorem is a straightforward calculation. Differentiate the identity (27.5) to get

$$\frac{dM}{da} = \frac{\partial g}{\partial x_1}\frac{dx_1}{da} + \frac{\partial g}{\partial x_2}\frac{dx_2}{da} + \frac{\partial g}{\partial a},$$

and substitute from the first-order conditions (27.4) to find

$$\frac{dM}{da} = \lambda \left[\frac{\partial h}{\partial x_1}\frac{dx_1}{da} + \frac{\partial h}{\partial x_2}\frac{dx_2}{da}\right] + \frac{\partial g}{\partial a}. \tag{27.6}$$

Now observe that the optimal choice functions must identically satisfy the constraint $h(x_1(a), x_2(a), a) \equiv 0$. Differentiating this identity with respect to a, we have

$$\frac{\partial h}{\partial x_1}\frac{dx_1}{da} + \frac{\partial h}{\partial x_2}\frac{dx_2}{da} + \frac{\partial h}{\partial a} \equiv 0. \tag{27.7}$$

Substitute (27.7) into (27.6) to find

$$\frac{dM}{da} = -\lambda \frac{\partial h}{\partial a} + \frac{\partial g}{\partial a},$$

which is the required result.