Math 13 Fall 2009: Final Exam
December 19, 2009

Instructions: There are 8 questions on this exam for a total of 100 points. You may not use any outside materials (e.g., notes, calculators, or other devices). Please turn off your cell phone. You have 3 hours to complete this exam. Remember to fully justify your answers.

Problem 1 (15 Points)**.**

(1) Find a vector function describing the line through $P = (2, 5, 7)$ and $Q = (4, 3, 8)$.
(2) Find the equation of the plane through the point $P = (2, 1, 5)$ and containing the line described by $\mathbf{r}(t) = \langle 3t, 2 + t, 2 - t\rangle$.
(3) Determine if the following two lines are parallel, intersecting, or skew:

$$x = 5 + 2t \quad y = -2 - 3t \quad z = 3 + t$$

and

$$x = 3 + 2s \quad y = -1 - 5s \quad z = 2 + s$$

Proof.

(1) We have the direction of the line as $\langle 2, -2, 1\rangle$ so we have vector function

$$\langle 2t + 2, -2t + 5, t + 7\rangle .$$

(2) We have two vectors in the plane $\langle 3, 1, -1\rangle$ and $\langle -2, 1, -3\rangle$. Taking their cross product gets the normal vector

$$\vec{n} = det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -1 \\ -2 & 1 & -3 \end{pmatrix} = \langle -2, 11, 5\rangle$$

so we have equation of the plane

$$-2(x - 2) + 11(y - 1) + 5(z - 5) = 0.$$

(3) The directions of the lines are $\langle 2, -3, 1\rangle$ and $\langle 2, -5, 1\rangle$ so they are not parallel. Solving for an intersection point we find that they intersect at $(1, 4, 1)$.

□

Problem 2 (16 Points)**.** Find the unit tangent vector $\mathbf{T}(t)$, unit normal vector $\mathbf{N}(t)$, unit binormal vector $\mathbf{B}(t)$, and curvature κ for the helix $\mathbf{r}(t) = (a \cos t)\hat{\imath} + (a \sin t)\hat{\jmath} + bt\hat{k}$, where $a, b \geq 0$.

Proof. We have

$$\begin{aligned}
\mathbf{r}'(t) &= (-a \sin t)\hat{\imath} + (a \cos t)\hat{\jmath} + b\hat{k} \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{(-a \sin t)\hat{\imath} + (a \cos t)\hat{\jmath} + b\hat{k}}{\sqrt{a^2 + b^2}} \\
\mathbf{T}'(t) &= ((-a \cos t)\hat{\imath} + (-a \sin t)\hat{\jmath} + 0\hat{k})/\sqrt{a^2 + b^2} \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = (-\cos t)\hat{\imath} + (-\sin t)\hat{\jmath} \\
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) = \frac{(b \sin t)\hat{\imath} + (-b \cos t)\hat{\jmath} + a\hat{k}}{\sqrt{a^2 + b^2}} \\
\kappa &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{a}{\sqrt{a^2+b^2}}}{\sqrt{a^2 + b^2}} = \frac{a}{a^2 + b^2}.
\end{aligned}$$

□

Problem 3 (12 Points)**.**

(1) Show that

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous at the origin.

(2) Show that

$$f(x,y) = \begin{cases} \frac{2x^2y+3y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous at the origin and calculate $\frac{\partial f}{dx}(0,0)$ and $\frac{\partial f}{dy}(0,0)$.

Proof.

(1) The function is a rational function so is continuous wherever it is defined. So we only need to check continuity at $(0,0)$. We show that the $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and hence the function is not continuous at $(0,0)$.

We first examine the path $x = y$ to get

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^3}{x^2+x^4} = 0.$$

and next the path $y = x^2$ to get

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^4}{x^4+x^4} = 1.$$

Since these values are not equal, then limit does not exist.

(2) The function is a rational function so is continuous wherever it is defined. So we only need to check continuity at $(0,0)$. Use the squeeze theorem to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. In particular,

$$\left|\frac{2x^2y+3y^3}{x^2+y^2}\right| \leq \left|\frac{2x^2y}{x^2+y^2}\right| + \left|\frac{3y^3}{x^2+y^2}\right| \leq \left|\frac{2x^2y}{x^2}\right| + \left|\frac{3y^3}{y^2}\right| = |2y| + |3y|$$

The $|2y| + |3y|$ goes to 0 and y goes to 0, so by the squeeze theorem the limit is 0.

We use the limit definition to compute

$$\frac{\partial f}{dx}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$\frac{\partial f}{dy}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{3h-0}{h} = 3.$$

□

Problem 4 (12 Points)**.**

(1) Let $f(x,y) = \sqrt{x^2+y^2}$. Find the tangent plane to the surface $z = f(x,y)$ at $(3,-4,5)$. Linearly approximate the value of $f(3.1,-4.1)$.

(2) Find the directional derivative of $f(x,y) = 2e^x \sin y$ at $(0,\pi/4)$ in the direction of $v = \langle 1,-1\rangle$. In what direction is the maximal rate of change at $(0,\pi/4)$?

Proof.

(1) For a surface $z = f(x,y)$ the normal vector to the tangent plane is $\langle f_x, f_y, -1\rangle$. We compute

$$\vec{n} = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle = \left\langle \frac{3}{5}, \frac{-4}{5}, -1 \right\rangle$$

which is in the same direction as $\langle 3,-4,5\rangle$. So we get tangent plane

$$3(x-3) - 4(y+4) - 5(z-5) = 0.$$

Linear approximation is given by

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

We compute

$$f(x,y) \approx 5 + \frac{3}{5}(x-3) - \frac{4}{5}(y+4)$$

and so

$$f(3.1,-4.1) \approx 5 + \frac{3}{5}\cdot\frac{1}{10} + \frac{4}{5}\cdot\frac{1}{10} = 5\frac{7}{50}.$$

(2) We compute the gradient

$$\nabla(f) = \langle 2e^x \sin y, 2e^x \cos y\rangle$$

at $(0,\pi/4)$ this is

$$\left\langle \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right\rangle = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$$

the direction is

$$u = \frac{v}{|v|} = \frac{\langle 1,-1\rangle}{\sqrt{2}}$$

so we get directional derivative

$$D_u f = \nabla f \cdot u = 1 - 1 = 0.$$

The maximal rate of change is in the direction of the gradient vector which is

$$\frac{\langle 1,1\rangle}{\sqrt{2}}.$$

□

Problem 5 (10 Points)**.** Find the absolute maximum and minimum values of $f(x,y) = x^2 + xy + y^2$ on the disk $x^2 + y^2 \leq 4$.

Proof. We first compute the critical points which are solutions to

$$f_x = 2x + y = 0 \quad \text{and} \quad f_y = x + 2y = 0$$

which has the critical point $(0,0)$ which is inside the disk.

Now computing the boundary condition with Lagrange Multipliers we need to solve

$$2x + y = \lambda 2x$$
$$x + 2y = \lambda 2y$$
$$x^2 + y^2 = 4$$

Multiplying the first by y and the second by x we have

$$2xy + y^2 = \lambda 2xy$$
$$x^2 + 2xy = \lambda 2yx$$
$$x^2 + y^2 = 4$$

So we have $2xy + y^2 = x^2 + 2x$ and hence $y^2 = x^2$ and so $y = \pm x$. Substituting into the constraint gives

$$x^2 + x^2 = 4$$

and so $x = \pm\sqrt{2}$. The 4 resulting points are $(\sqrt{2}, \pm\sqrt{2})$ and $(-\sqrt{2}, \pm\sqrt{2})$. To find the absolute max/min we check

$$f(0,0) = 0$$
$$f(\pm\sqrt{2}, \pm\sqrt{2}) = 2 + 2 + 2 = 6$$
$$f(\pm\sqrt{2}, \mp\sqrt{2}) = 2 - 2 + 2 = 2$$

So the absolute min is 0 which occurs at $(0,0)$ and the absolute max is 6 which occurs at $(\sqrt{2},\sqrt{2})$ and $(-\sqrt{2},-\sqrt{2})$. □

Problem 6 (10 Points)**.** Find the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$ and inside the sphere $x^2 + y^2 + z^2 = 25$.

Proof. By symmetry we compute the volume about $z = 0$ and double it. We have

$$V = 2\int_0^{2\pi}\int_0^4\int_0^{\sqrt{25-r^2}} r\ dz\ dr\ d\theta = 4\pi\int_0^4 r\sqrt{25-r^2}dr = \frac{392\pi}{3}.$$

□

Problem 7 (10 Points)**.** Evaluate

$$\iint_D \frac{x-y}{x+y}\ dA$$

using the change of variables $u = x + y$ and $v = x - y$, where D is the region bounded by the lines $y = x$, $y = x - 1$, $y = 1 - x$, and $y = 2 - x$.

Proof. Rearranging the four boundaries we see that

$$y - x = 0, y - x = -1 \quad \text{and} \quad y + x = 1, y + x = 2.$$

Therefore, the new region is bounded by $v = 0$, $v = 1$, $u = 1$, and $u = 2$. We solve for x, y interms of u, v as

$$x = \frac{u+v}{2} \quad \text{and} \quad y = \frac{u-v}{2}.$$

We compute the jacobian as

$$\left|\det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}\right| = \left|-\frac{1}{4} - \frac{1}{4}\right| = \frac{1}{2}.$$

Applying the change of variables we have

$$\iint_D \frac{x-y}{x+y}\ dA = \frac{1}{2}\int_0^1\int_1^2 \frac{v}{u}\ du\ dv = \frac{1}{2}\ln 2\int_0^1 vdv = \frac{1}{4}\ln 2.$$

□

Problem 8 (15 Points)**.**

(1) Calculate the line integral $\int_C (e^y + ye^x)dx + (e^x + xe^y)dy$, where C is a path that begins at (0,0) and ends at (1,-1).
(2) Evaluate the line integral $\int_C (y + e^x)dx + (2x^2 + \cos y)dy$, where C is the boundary of the triangle with vertices (0,0), (1,1), and (2,0) traversed once counterclockwise.
(3) Calculate $\int_C f ds$ where $f(x, y) = x + y$ and C is the curve $x^2 + y^2 = 4$ in the first quadrant from $(2, 0)$ to $(0, 2)$.

Proof.

(1) We compute the partial integrals

$$\int e^y + ye^x dx = xe^y + ye^c + C(y)$$
$$\int e^x + xe^y dy = ye^x + xe^y + C(x).$$

Then we can choose the potential function $f(x, y) = xe^y + ye^x$, so

$$\int_C (e^y + ye^x)dx + (e^x + xe^y)dy = f(1, -1) - f(0, 0) = e^{-1} - e.$$

(2) The region is a close, piecewise smooth, simple region in positive orientation, so we can apply Green's Theorem to get

$$\int_C (y + e^x)dx + (2x^2 + \cos y)dy = \int_0^1\int_y^{2-y} (4x - 1)dxdy = 3.$$

(3) We parameterize the quarter circle as $r(t) = \langle 2\sin t, 2\cos t\rangle$, $0 \le t \le \pi/2$. Then ds is given by

$$ds = \sqrt{(-2\sin t)^2 + (2\cos t)^2}dt = 2dt.$$

Now we integrate

$$\int_C f\ ds = \int_0^{\pi/2} (2\cos t + 2\sin t)2dt = 8.$$

□