Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of January 31, 2014

There are 9 problems (10 points each, totaling 90 points) on this portion of the examination. **Show all of your work.**

1. Find the critical points of the function $f(x, y) = x^4 - 4xy + 2y^2$ and classify as a local maximum, local minimum, or a saddle point.

   Solution: Since f is a polynomial, it is differentiable on $\mathbb{R}^2$. The critical points occur when
   $$f_x(x, y) = 4x^3 - 4y = 0 \qquad \text{and} \qquad f_y(x, y) = -4x + 4y = 0.$$
   The second equation gives $x = y$, and substituting it into the first gives $4x^3 - 4x = 0$, or $x(x + 1)(x - 1) = 0$. Thus $x = 0$ or $x = \pm 1$. Therefore the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

   To classify of the critical points, we use the second derivative test. First let us compute the second derivatives:
   $$f_{xx}(x, y) = 12x^2 \qquad f_{xy}(x, y) = -4 \qquad f_{yy}(x, y) = 4$$
   $$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 = 48x^2 - 16$$
   $D(0, 0) = -16 < 0$, so $(0, 0)$ is a saddle point; $D(1, 1) = 32 > 0$ and $f_{xx}(1, 1) = 12 > 0$, so $(1, 1)$ is a local minimum; and $D(-1, -1) = 32 > 0$, $f_{xx}(-1, -1) = 12 > 0$, so $(-1, -1)$ is also a local minimum.

2. Suppose the plane $z = 2x - y - 1$ is tangent to the graph of $z = f(x, y)$ at $P = (5, 3)$.

   (a) Determine $f(5, 3)$, $\frac{\partial f}{\partial x}(5, 3)$ and $\frac{\partial f}{\partial x}(5, 3)$.
   Solution: We know the graphs of $z = 2x - y - 1$ and $z = f(x, y)$ intersect at $P(5, 3, f(5, 3))$, so $f(5, 3) = 2(5) - 3 - 1 = 6$. Furthermore, recall that an equation of the tangent plane to $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is
   $$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$
   So we have
   $$z = 2x - y - 1 = 6 + \frac{\partial f}{\partial x}(5, 3)(x - 5) + \frac{\partial f}{\partial y}(5, 3)(y - 3).$$
   Comparing coefficients of x and y, we obtain
   $$\frac{\partial f}{\partial x}(5, 3) = 2 \qquad \frac{\partial f}{\partial y}(5, 3) = -1.$$

   (b) Estimate $f(5.2, 2.9)$.
   Solution: We use the linear approximation of f at $(5, 3)$:
   $$f(5.2, 2.9) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$\approx f(5,3) + \tfrac{\partial f}{\partial x}(5,3)(5.2-5) + \tfrac{\partial f}{\partial y}(5,3)(2.9-3)$$
$$\approx 6 + 2\cdot(.2) + (-1)\cdot(-.1) = 6.5.$$

Here is another way to do this. Near $(5,3)$, the graph $z = f(x,y)$ is approximated by the tangent plane at $(5,3)$, which is given as $z = 2x - y - 1$. Thus

$$f(5.2, 2.9) \approx 2(5.2) - (2.9) - 1 = 6.5.$$

3. Calculate the volume of the region **inside** sphere $x^2 + y^2 + z^2 = a^2$ and **outside** the cylinder $x^2 + y^2 = b^2$, where $a > b$, by using an appropriate double integral.

Solution: We are removing a vertical cylinder of radius b from a sphere of radius a. When we think of this as double integral, the region in the plane is

$$R = \{(x,y) \mid b \le \sqrt{x^2+y^2} \le a\} = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ b \le r \le a\}.$$

The "top" of the figure is the top half of the sphere, given by $z = \sqrt{a^2 - r^2}$, and the "bottom" is the bottom half of the sphere, given by $z = -\sqrt{a^2 - r^2}$. Here, we are using cylindrical coordinates. Then the double integral giving the volume is

$$\begin{aligned} V &= \iint_R \sqrt{a^2 - r^2} - (-\sqrt{a^2 - r^2})\, dA \\ &= \int_0^{2\pi} \int_b^a 2r\sqrt{a^2 - r^2}\, dr\, d\theta \\ &= 2\pi \left[-\frac{2}{3}\left(a^2 - r^2\right)^{3/2} \right]_b^a \\ &= -\frac{4\pi}{3}\left((a^2 - a^2)^{3/2} - (a^2 - b^2)^{3/2}\right) \\ &= \frac{4\pi}{3}\left(a^2 - b^2\right)^{3/2}. \end{aligned}$$

4. Suppose that $\mathbf{r}(t) = (3\sqrt{2}t, e^{-3t}, e^{3t})$ describes the position of an object at time t.

   (a) Calculate the acceleration of the object at time t.
   Solution: The velocity and acceleration of the object at time t are

$$\mathbf{v}(t) = \mathbf{r}'(t) = (3\sqrt{2}, -3e^{-3t}, 3e^{3t})$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = (0, 9e^{-3t}, 9e^{3t}).$$

   (b) Calculate the speed of the object at time t. Simplify by factoring the expression under the square root.
   Solution: The speed of the object at time t is

$$\begin{aligned} |\mathbf{v}(t)| &= \sqrt{(3\sqrt{2})^2 + (-3e^{-3t})^2 + (3e^{3t})^2} \\ &= \sqrt{18 + 9e^{-6t} + 9e^{6t}} = \sqrt{9(e^{6t} + 2 + e^{-6t})} \\ &= 3\sqrt{(e^{3t} + e^{-3t})^2} = 3(e^{3t} + e^{-3t}). \end{aligned}$$

(c) Calculate the distance traveled by the object between times $t = 0$ and $f = 1$.

Solution: The total distance traveled by the object between $t = 0$ and $t = 1$ is

$$D = \int_0^1 |\mathbf{v}(t)|\,dt = \int_0^1 3e^{3t} + 3e^{-3t}\,dt = e^{3t} - e^{-3t}\Big|_0^1 = e^3 - \frac{1}{e^3}.$$

5. Consider the function

$$f(x,y) = \begin{cases} \dfrac{3y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

Solution: Recall that f is continuous at $(0,0)$ iff $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Hence we must show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3y^3}{x^2+y^2} = 0$. In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $(x,y) \to 0$ becomes $r \to 0$. Then the limit is

$$\lim_{(x,y)\to(0,0)} \frac{3y^3}{x^2+y^2} = \lim_{r\to 0} \frac{3r^3\sin^3\theta}{r^2} = \lim_{r\to 0} 3r\sin^3\theta = 0,$$

where the last equality follows since $3r \to 0$ and $\sin^3\theta$ is bounded.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

Solution: We will directly apply the definition of partial derivatives.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{0}{h^3} = 0$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{3h^3}{0+h^2} - 0}{h} = \lim_{h\to 0} \frac{3h}{h} = 3.$$

6. Suppose $T : V \to V$ is a linear transformation, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for V, and the matrix representation of T with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & -3 \\ -1 & 19 & 0 \end{bmatrix}.$$

Determine $T(2\mathbf{b}_1 + 4\mathbf{b}_3)$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.

Solution: The coordinate vector of $2\mathbf{b}_1 + 4\mathbf{b}_3$ relative to $\mathcal{B}$ is $\mathbf{a} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$. Then the coordinate vector of $T(2\mathbf{b}_1 + 4\mathbf{b}_3)$ relative to $\mathcal{B}$ is

$$\mathbf{b} = A\mathbf{a} = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & -3 \\ -1 & 19 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ -2 \end{bmatrix}.$$

Hence

$$T(2\mathbf{b}_1 + 4\mathbf{b}_3) = 24\mathbf{b}_1 + 2\mathbf{b}_2 - 2\mathbf{b}_3.$$

7. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix}$.

(a) Compute the eigenvalue(s) of A.

Solution: The eigenvalues of A are the roots of its characteristic polynomial

$$\det(A - \lambda I_3) = \det\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 2 & 6-\lambda & 2 \\ 3 & 0 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)\det \begin{bmatrix} 6-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = (2-\lambda)(6-\lambda)(1-\lambda).$$

Thus A has three eigenvalues: $\lambda = 1$, $\lambda = 2$, and $\lambda = 6$.

(b) Find an invertible matrix C such that $C^{-1}AC$ is diagonal.

Solution: We must first compute bases of the eigenspaces corresponding to the eigenvalues of A.

$\lambda = 1$: The eigenspace is the solution space of $(A - I)\,\vec{x} = \vec{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 2 \\ 3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/5 \\ 0 & 0 & 0 \end{bmatrix},$$

so we have $x_1 = x_2 + (2/5)x_3 = 0$ with free variable x_3. The solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (-2/5)x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2/5 \\ 1 \end{bmatrix}.$$

Thus the eigenspace of A associated to $\lambda = 1$ has basis $\left\{ \begin{bmatrix} 0 \\ -2/5 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 2$: The eigenspace is the solution space of $(A - 2I)\,\vec{x} = \vec{0}$. Being homogeneous, we row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 2 \\ 3 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix},$$

so we have $x_1 - (1/3)x_3 = x_2 + (2/3)x_3 = 0$ with free variable x_3. The solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/3)x_3 \\ (2/3)x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix}.$$

Thus the eigenspace of A associated to $\lambda = 2$ has basis $\left\{ \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 6$: The eigenspace is the solution space of $(A - 6I)\,\vec{x} = \vec{0}$. As above, we row reduce the coefficient matrix:

$$\begin{bmatrix} -4 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so we have $x_1 = x_3 = 0$ with free variable x_2. The solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the eigenspace of A associated to $\lambda = 6$ has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Finally, because A has three linearly independent eigenvectors, it is diagonalizable (the problem implies this anyway). Its eigenvectors form the columns of the desired matrix

$$C = \begin{bmatrix} 0 & 1/3 & 0 \\ -2/5 & -2/3 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(*Note:* One can check that $C^{-1}AC$ is diagonal, but doing so is not recommended because the calculations are extremely time-consuming.)

8. Let $A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & 2 \\ 2 & -2 & 4 & \alpha \end{bmatrix}$, where α is a real number.

   (a) For what values of α does $A\mathbf{x} = \mathbf{b}$ have at least one solution for all $\mathbf{b} \in \mathbb{R}^3$?

   Solution: Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. We can row reduce to obtain

$$\begin{bmatrix} 1 & -1 & 1 & 3 & b_1 \\ -1 & 1 & 0 & 2 & b_2 \\ 2 & -2 & 4 & \alpha & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 3 & b_1 \\ 0 & 0 & 1 & & b_1 + b_2 \\ 0 & 0 & 2 & \alpha - 6 & b_3 - 2b_1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 & -b_2 \\ 0 & 0 & 1 & 1 & b_1 + b_2 \\ 0 & 0 & 0 & \alpha - 16 & b_3 - 4b_1 - 2b_2 \end{bmatrix}$$

   To finish the row reduction, there are two cases, depending on $\alpha - 8$:

   $\boxed{\alpha - 16 \neq 0}$ We can divide the last row by $\alpha - 16$ to obtain a leading 1 in the 4th column. Hence we have at least one solution, no matter how we choose $\mathbf{b} \in \mathbb{R}^3$.

   $\boxed{\alpha - 16 = 0}$ If we pick $\mathbf{b} \in \mathbb{R}^3$ so that $b_3 - 4b_1 - 2b_2 \neq 0$, then the last row will have a leading 1 in the last column of the augmented matrix, which implies that the system has no solution. So $\mathbf{b} \in \mathbb{R}^3$ can be chosen so that there is no solution.

   $\boxed{\text{Conclusion}}$ $A\mathbf{x} = \mathbf{b}$ has at least one solution for all $\mathbf{b} \in \mathbb{R}^3$ if and only if $\alpha \neq 16$.

(b) For the remainder of the problem set $\alpha = 11$. Find the general solution of $A\mathbf{x} = \mathbf{0}$.

Solution: Set $\alpha = 11$ in the partial row reduction of part (a). If we focus on the coefficient matrix ($A\mathbf{x} = \mathbf{0}$ is homogeneous), we get the row reduction

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & 2 \\ 2 & -2 & 4 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 11-8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which gives the equations $x_1 - x_2 = x_3 = x_4 = 0$ with free variable x_2. Thus the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

9. Suppose $\{u, v\}$ is a basis for a vector space V. Prove that $\{u + 2v, 3u - v\}$ is also a basis for V.

Solution: Our hypothesis implies that V has dimension 2, and we are asked to prove that $\{u + 2v, 3u - v\}$ is a basis for V Recall that n vectors in an n-dimensional vector basis form a basis $\iff$ they span $\iff$ they are linearly independent. Here, it is easier to show that $u+2v, 3u-v$ are linearly independent. Suppose $a\,(u+2v)+b\,(3u-v) = 0$, where a, $b \in \mathbb{R}$. Then

$$\begin{aligned} au + 2av + 3bu - bv &= 0 \\ \Rightarrow \ (a + 3b)u + (2a - b)v &= 0. \end{aligned}$$

Since $\{u, v\}$ is a basis for V, we know that u and v are linearly independent. Therefore the last equation implies that

$$a + 3b = 0 \qquad \text{and} \qquad 2a - b = 0.$$

It is easy to show that the only solution to the above system of equations is $a = b = 0$, which in turn implies that $u+2v$ and $3u-v$ are linearly independent. As noted above, V has dimension 2, so that any set of 2 linearly independent vectors in V is a basis. Therefore, $\{u + 2v,\ 3u - v\}$ is a basis for V.