Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of January 29, 2016

1. [25 points] Find two points on the ellipsoid $x^2 + 2y^2 + 4z^2 = 10$ where the tangent plane is perpendicular to the vector $\langle 1,\, 1,\, -2\rangle$.

   Solution: The tangent plane to $f(x, y, z) = x^2 + 2y^2 + 4z^2 = 10$ has normal vector $\nabla f(x, y, z) = \langle 2x, 4y, 8z\rangle$. So the tangent plane is perpendicular to $\langle 1,\, 1,\, -2\rangle$ when the gradient is *parallel* to $\nabla f(x, y, z)$. Thus we have the equation $\nabla f(x, y, z) = \lambda\langle 1,\, 1,\, -2\rangle$, which implies
   $$2x = \lambda, \quad 4y = \lambda, \quad 8z = -2\lambda.$$
   These equations yield $x = \frac{1}{2}\lambda$, $y = \frac{1}{4}\lambda$ and $z = -\frac{1}{4}\lambda$. Hence $y = \frac{1}{2}x$ and $z = -\frac{1}{2}x$. Substituting into $x^2 + 2y^2 + 4z^2 = 10$, we obtain
   $$10 = x^2 + 2(\tfrac{1}{2}x)^2 + 4(-\tfrac{1}{2}x)^2 = x^2 + \tfrac{1}{2}x^2 + x^2 = \tfrac{5}{2}x^2,$$
   so that $x^2 = \frac{2}{5}\cdot 10 = 4$. Thus $x = \pm 2$, which when combined with $y = \frac{1}{2}x$ and $z = -\frac{1}{2}x$ gives the desired points
   $$(x, y, z) = \pm(2, 1, -1).$$

2. [25 points] Let $f(x,y) = \begin{cases} \dfrac{4x^2 + 3xy + 2y^2}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 2 & \text{if } (x, y) = (0, 0). \end{cases}$

   (a) [15 points] Compute $f_x(0, 0)$ and $f_y(0, 0)$.
   Solution: First observe that when $h \neq 0$, we have
   $$f(h, 0) = \frac{4h^2 + 0 + 0}{2h^2 + 0} = 2, \quad f(0, h) = \frac{0 + 0 + 2h^2}{0 + h^2} = 2.$$
   Then we use the definition of partial derivative to obtain
   $$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{2 - 2}{h} = \lim_{h\to 0} 0 = 0$$
   and
   $$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{2 - 2}{h} = \lim_{h\to 0} 0 = 0.$$

   (b) [10 points] Prove that f is **not** continuous at $(0, 0)$.
   Solution: Recall that f is continuous at $(0, 0)$ iff $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$. Therefore we must show that the limit does not equal 2. Let $(x, y) \to (0, 0)$ along $y = mx$, where m is any real number. Then for $x \neq 0$,
   $$f(x, y) = f(x, mx) = \frac{4x^2 + 3x(mx) + 2(mx)^2}{2x^2 + (mx)^2} = \frac{x^2(4 + 3m + 2m^2)}{x^2(2 + m^2)} = \frac{4 + 3m + 2m^2}{2 + m^2}.$$
   Thus $f(x, y) \to (4 + 3m + 2m^2)/(2 + m^2)$ as $(x, y) \to (0, 0)$ along $y = mx$, so f has different limits along different paths. Therefore $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, and f is not continuous at $(0, 0)$.

3. [25 points] Find the points at which the absolute maximum and minimum of the function $f(x,y) = xy - 1$ on the disk $x^2 + y^2 \leq 2$ occur. State all points where the extrema occur as well as the maximum and minimum values.

   Solution: We know that there exist both an absolute maximum and minimum because f is continuous on the closed and bounded disk. The first step is thus to find the critical points of f in the disk, which occur when $f_x = y = 0$ and $f_y = x = 0$, or at $(x,y) = (0,0)$.

   We next use Lagrange multipliers to find where the extreme values of f can occur on the boundary of the disk, $x^2 + y^2 = 2$. Writing this as $g(x,y) = x^2 + y^2 - 2 = 0$, Lagrange multipliers gives the equations

   $$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = 0,$$

   where can be written as

   $$y = \lambda \cdot 2x, \quad x = \lambda \cdot 2y, \quad x^2 + y^2 - 2 = 0.$$

   The first two equations imply $y = 4\lambda^2 y$, so $y(4\lambda^2 - 1) = 0$. But $y = 0$ would imply $x = 0$, which doesn't satisfy $g(x,y) = 0$. Hence $4\lambda^2 - 1 = 0$, so $2\lambda = \pm 1$. It follows that $y = \pm x$. Substituting into the constraint gives $2x^2 = 2$, so $x = \pm 1$. Thus we get the four boundary points $(\pm 1, \pm 1)$.

   In conclusion, there are five points on the disk where extrema could possibly occur: $(0,0)$, $(\pm 1, \pm 1)$. Since $f(0,0) = -1$, $f(1,1) = f(-1,-1) = 0$, and $f(1,-1) = f(-1,1) = -2$, we conclude that the absolute maximum occurs at $(1,1)$ and $(-1,-1)$ and has a value of 0, while the absolute minimum occurs at $(1,-1)$ and $(-1,1)$ and has a value of -2.

4. [25 points] Consider the paraboloid $z = x^2 + y^2$, which is intersected by the plane $z = 4$.

   (a) [15 points] Find the volume of the region that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$.

   Solution: In cylindrical coordinates the paraboloid is $z = r^2$, and where the paraboloid intersects the plane, $r^2 = x^2 + y^2 = z = 4$, so we may express the region as

   $$E = \{(r,\, \theta,\, z) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2,\ r^2 \leq z \leq 4\}.$$

   Therefore the volume is

   $$\begin{aligned} V = \iiint_E dV &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\, dz\, dr\, d\theta \\ &= 2\pi \int_0^2 \big[rz\big]_{r^2}^4\, dr = 2\pi \int_0^2 4r - r^3\, dr \\ &= 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2 = 2\pi\Big(\Big(2 \cdot 2^2 - \frac{2^4}{4}\Big) - 0\Big) \\ &= 2\pi(8 - 4) = 8\pi. \end{aligned}$$

(b) [10 points] Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that is below the plane $z = 4$.

Solution: The given surface lies above the disk D in the xy-plane where $x^2+y^2 \leq 4$. Therefore the surface area is

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA \\ &= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA. \end{aligned}$$

Converting to polar coordinates, we have

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= 2\pi \int_0^2 \frac{1}{8}(1 + 4r^2)^{1/2}\, 8r\, dr, \quad u = 1 + 4r^2,\ du = 8r\, dr \\ &= 2\pi \int_1^{17} \frac{1}{8}\, u^{1/2}\, du = \frac{\pi}{4}\left[\frac{2}{3}(u)^{3/2}\right]_1^{17} \\ &= \frac{\pi}{6}\left(17\sqrt{17} - 1\right). \end{aligned}$$

5. [25 points] Let V be the vector space of polynomials of degree at most 2 and let $T : V \to V$ be the mapping given by

$$T(ax^2 + bx + c) = (b - a)x^2 + (c - b)x + (a - c).$$

(a) [10 points] Prove that T is a linear transformation.

Solution: We first show that T preserves addition. Let $\mathbf{u} = a_1x^2 + b_1x + c_1, \mathbf{v} = a_2x^2 + b_2x + c_2 \in V$, where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$. Then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T((a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)) \\ &= T((a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)) \\ &= (b_1 + b_2 - (a_1 + a_2))x^2 + (c_1 + c_2 - (b_1 + b_2))x + (a_1 + a_2 - (c_1 + c_2)) \\ &= ((b_1 - a_1)x^2 + (c_1 - b_1)x + (a_1 - c_1)) + ((b_2 - a_2)x^2 + (c_2 - b_2)x + (a_2 - c_2)) \\ &= T(a_1x^2 + b_1x + c_1) + T(a_2x^2 + b_2x + c_2) = T(\mathbf{u}) + T(\mathbf{v}). \end{aligned}$$

We now show T preserves scalar multiplication. Let k be a scalar.

$$\begin{aligned} T(k\mathbf{u}) &= T(k(a_1x^2 + b_1x + c_1)) = T(a_1kx^2 + b_1kx + c_1k)) \\ &= (b_1k - a_1k)x^2 + (c_1k - b_1k)x + (a_1k - c_1k) \\ &= k((b_1 - a_1)x^2 + (c_1 - b_1)x + (a_1 - c_1)) \\ &= kT(a_1x^2 + b_1x + c_1) = kT(\mathbf{u}). \end{aligned}$$

Thus T is linear by definition.

(b) [10 points] Calculate the dimension of the null space, or kernel, of T.
Solution: The kernel $\ker T$ is the subset of V consisting of all polynomials that map to the zero polynomial $0x^2 + 0x + 0$. We see that

$$T(ax^2 + bx + c) = (b - a)x^2 + (c - b)x + (a - c) = 0x^2 + 0x + 0$$

if and only if $b - a = c - b = a - c = 0$, which is equivalent to $a = b = c$. Thus $\ker T$ is the set of all vectors of the form $kx^2 + kx + k$, where $k \in \mathbb{R}$. In other words, $\ker T = \text{Span}\,(\{x^2 + x + 1\})$. It therefore has dimension 1.

(c) [5 points] Calculate the dimension of the image space, or range, of T.
Solution: By the rank/nullity theorem, $\dim \ker T + \dim \text{range } T = \dim \text{ domain } T$. We have already established in part (b) that $\dim \ker T = 1$. Because $V = \text{Span }(\{x^2, x, 1\})$, $\dim \text{ domain } T = 3$. Thus, $\dim \text{ range } T = 3 - 1 = 2$.

6. [30 points]

(a) [15 points] Find a basis for the subspace of $\mathbb{R}^3$ given by the span of the following set of vectors:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Solution: For easier reference, let us call the given set of vectors S. Since we know that a basis for a subspace of $\mathbb{R}^3$ consists of at most three vectors, the vectors in S are linearly dependent. Our task is to find a subset of S that forms a basis for Span$\,(S)$. We first express the four vectors in matrix form and calculate its reduced echelon form:

$$\begin{bmatrix} 2 & -4 & 1 & 0 \\ -1 & 2 & -2 & 3 \\ 1 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 2 & -2 & 3 \\ 2 & -4 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are pivots (leading 1s) in columns 1 and 3, we conclude that the first and the third vectors in S form a basis of the span. The desired basis is then

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

(b) [15 points] Give an example of a vector in $\mathbb{R}^3$ that is not in the subspace in part (a). Justify your answer.
Solution: Any vector in $\mathbb{R}^3$ that cannot be expressed as a linear combination of the two vectors in $\mathcal{B}$ is acceptable as an answer. An easy method of finding one such vector is to take the cross product of the two vectors, as the result is orthogonal to both vectors and thus cannot be in their span. In this case,

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

7. [20 points] Suppose that V is a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Prove that the set

$$\{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1\}$$

is also a basis for V.

Solution: Because V has as a basis consisting of three vectors, it has dimension 3. Recall that n vectors in an n-dimensional vector basis form a basis $\iff$ they span $\iff$ they are linearly independent. Here, it is easier to show that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1$ are linearly independent. Suppose that

$$a(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + b(\mathbf{v}_1 + \mathbf{v}_2) + c\mathbf{v}_1 = \mathbf{0}.$$

Then

$$(a + b + c)\mathbf{v}_1 + (a + b)\mathbf{v}_2 + a\mathbf{v}_3 = \mathbf{0},$$

which implies

$$a + b + c = a + b = a = 0$$

since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. These equations imply $a = 0$ and then $b = 0$ and finally $c = 0$. We conclude that $\{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1\}$ is linearly independent and hence is a basis by the theorem noted above.

8. [25 points]

(a) [10 points] Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution: The eigenvalues of A are the roots of its characteristic polynomial

$$\begin{aligned}
\det(A - \lambda I_3) &= \det\left(\begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} -\lambda & 4 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & -1 & -\lambda \end{bmatrix} \\
&= -\lambda \det\begin{bmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = -\lambda\big((-2-\lambda)(-\lambda) + 1\big) \\
&= -\lambda(\lambda^2 + 2\lambda + 1) = -\lambda(\lambda+1)^2.
\end{aligned}$$

This has two roots: $\lambda = 0$ and $\lambda = -1$. These are the two eigenvalues of A.

(b) [15 points] Show that the matrix A is not diagonalizable.

Solution: We know that an 3×3 matrix is diagonalizable iff it has 3 linearly independent eigenvectors. To prove that A is not diagonalizable, we will examine the eigenspaces.

$\lambda = -1$. The eigenspace is the solution space of $(A+I_3)\mathbf{x} = \mathbf{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of -1) is $3 - 2 = 1$. Hence there is only one linearly independent eigenvector when $\lambda = -1$.

$\lambda = 0$. The eigenspace is the solution space of $(A - 0I_3)\mathbf{x} = A\mathbf{x} = \mathbf{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of 0) is $3 - 2 = 1$. Hence there is only one linearly independent eigenvector when $\lambda = 0$.

It follows that there are only two linearly independent eigenvectors. So A is not diagonalizable.

Alternate Solution: The 3×3 matrix A is diagonalizable if and only if A has three linearly independent eigenvectors. The theory of eigenspaces provides the tools needed to decide whether or not A is diagonalizable:

- The dimension of each eigenspace is bounded by the multiplicity of the corresponding eigenvalue in the characteristic polynomial. Thus the eigenspace of $\lambda = 0$ has dimension ≤ 1 and the eigenspace of $\lambda = -1$ has dimension ≤ 2.
- A is diagonalizable $\iff$ the dimensions of the eigenspaces add up to 3 $\iff$ if the dimension of the each eigenspace *equals* the multiplicity of the eigenvalue in the characteristic polynomial.

We begin with $\lambda = -1$. The eigenspace is the solution space of $(A + I_3)\mathbf{x} = \mathbf{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of -1) is $3 - 2 = 1$. This is strictly smaller than 2, which is the multiplicity of -1 as a root of the characteristic polynomial. Hence A is not diagonalizable.