The Two-Price Mechanism in Non-Quasilinear Settings

Kevin Ma

April 29, 2023

Advisors: Christopher Kingston and Tyler Porter

Submitted to the Department of Economics of Amherst College in partial fulfillment of the requirements for the degree of Bachelor of Arts with honors

Abstract

This paper studies incentive-compatible redistributive mechanisms where agents do not have quasilinear preferences. Specifically, I focus on the two-price mechanism where consumers can choose between a high price at which trade is guaranteed and a low price at which the consumer has some probability of obtaining the good, like purchasing a raffle ticket. Because consumers do not have quasilinear preferences and experience wealth effects, redistribution becomes a crucial element of a utilitarian mechanism designer's goals. I use the two-price mechanism to screen relatively wealthier and poorer consumers and redistribute the profits from the mechanism. The main result of this paper is the competitive outcome does not always maximize total utility in this setting. Instead, the designer rations the poorer consumers to induce redistribution from the wealthier consumers, even when the good is plentiful. This leads to the counterintuitive result that an ex-post Pareto inefficient allocation of goods maximizes a utilitarian social welfare function in expectation. I begin with a motivating example of two consumers with the same valuation of the good but different wealth. Then, the model is extended to an infinite number of consumers with uniformly distributed wealth. Finally, I discuss how the mechanism can be implemented in the context of congestion pricing.

Keywords: mechanism design, wealth effects, redistribution

JEL Classification: D47, D61, D63, D82, H21, H23

Acknowledgements

Thank you Professor Kingston and Professor Porter for all your guidance and advice throughout the thesis process and for wading through all that math with me. You both helped me improve my economic thinking and writing and allowed me to push forward when I was stuck.

Thank you Professor Baisa for sparking my fascination with economic theory in Advanced Micro and Mechanism Design, and inspiring the topic of this thesis. Your perpetual kindness and support motivated me to do better throughout my time at Amherst.

Thank you Professor Ishii for having high standards for me and always pushing me to do better. You taught me that my model is not about the math but should tell a story and urged me to think about the intuition. I appreciate the thoughtful (and ample) comments on my work.

Contents

1 Introduction **1**

1.1 Related Literature 5

2 The Two Consumer Case: A Motivating Example **6**

2.1 Solving for Optimal Parameters 8

2.2 Interpretation of Optimal Parameters 10

2.3 Introducing Scarcity 15

2.4 Discussion 17

3 Model **18**

3.1 Analytical Results 19

3.2 Numerical Analysis 26

4 Policy Implications **35**

4.1 A Market with a Negative Externality 36

4.2 Congestion Pricing 38

5 Conclusion **40**

References **42**

Appendix **44**

1 Introduction

This paper tackles two main questions. How do traditional mechanism design results change after introducing wealth effects? And in this setting, how can redistribution be achieved in an incentive-compatible way? The canonical model of mechanism design assumes that agents have quasilinear utility and, therefore, the same marginal utility of wealth, making the distribution of wealth irrelevant to the mechanism designer's goals since all agents' utilities are transferable. However, this framework fails to capture the fact that people experience diminishing marginal utility of money in reality. Diminishing marginal utility is a widespread assumption in economics and has been confirmed empirically by Horowitz and McConnell (2007) in both price and exchange settings. This paper further investigates this question by modeling agents with wealth effects in a simple marketplace. With wealth effects, the redistribution of wealth now becomes a crucial element in obtaining efficiency.

The US federal government engages in redistributive policies through a system of progressive taxation and redistribution. However, this system suffers from cheating and tax fraud because wealthier individuals are not individually incentivized to participate in the system. The threat of law enforcement acts as a deterrent against tax evasion but ideally, individuals should participate in their own self-interest in the absence of a threat. So I consider how to achieve redistribution where incentive compatibility constraints are enforced.

The equity-efficiency tradeoff is often mentioned when referring to redistributive policies. The claim is that policies that increase equity by distorting the competitive equilibrium come at a cost to total productivity and efficiency. However, Stiglitz (2016) argues that excessive inequality hurts economic performance and that the US has certainly reached these extremes. He finds that high inequality leads to low aggregate demand, inequality of opportunity, and decreased productive public investments which decrease long-term economic growth. Because inequality has adverse effects on economic growth, decreasing inequality not only has positive social benefits but also economic ones. So he refutes the tradeoff between efficiency and equity, claiming it is illusory, at least with the inequality observed in the US today. This motivates the study of these redistributive mechanisms that work on reducing inequality.

This paper uses an adapted version of the two-price mechanism from Dworczak, Kominers, and Akbarpour (2021). They investigate a buyer-seller market where agents have varied and constant marginal utilities for money. On the buyer-side, they proved that the two-price mechanism is optimal among all incentive-compatible, individually-rational, budget-balanced, market-clearing mechanisms, which justifies the use of this mechanism in this paper. The varied marginal utilities for money make this problem non-quasilinear and enable welfare-enhancing redistribution. The two-price mechanism they presented consists of a high price at which trade is guaranteed and a low price at which the buyer has some probability of paying that price and obtaining the good. For reasons of analytical tractability, the mechanism is modified such that the consumer must pay the low price and then have some probability of obtaining the good, like purchasing a raffle ticket. If consumers experience wealth effects, this mechanism allows the mechanism designer to screen between relatively wealthier and poorer agents since they differ in their willingness to pay for the good.

The main intuition is that wealthier consumers have the highest willingness to pay so they are charged the high price and the profits from the mechanism are redistributed to all agents. In order to enforce incentive-compatibility constraints, the low price must become less attractive as the high price increases and so the poorer consumers are rationed. It can be argued that subjecting the poorest consumers to variance through this raffle ticket is unethical as they should be the most risk-averse. However, to preserve incentive-compatibility, it is impossible to guarantee the good to the poorest consumers while still charging the wealthier consumers a high price. Even though the poorest consumers do not receive the good with certainty, lowering their probability of obtaining the good increases the amount of redistribution through the wealthier consumers paying the high price. So the poorest consumers are compensated for their loss of the good through increased redistribution.

I analyze a buyer market where the designer supplies a plentiful good. The main result of this paper is that relative to giving the good away for free to all consumers, inducing redistribution by setting a positive price can increase total utility, even when it causes rationing in the market. This mirrors the result from Dworczak, Kominers, and Akbarpour but this paper extends their result into a more general setting and solves for the optimal parameters of the mechanism. They assumed that agents have constant marginal utilities of money while I provide agents with wealth and a functional form for utility that satisfies monotonicity and diminishing marginal utility in wealth. It is unclear whether the two-price mechanism is still optimal in this more general case but it is used here as a starting point for thinking about more general redistributive mechanisms. I imagine a continuous menu of prices and probabilities such that the incentive-compatibility constraints bind for all wealth levels could dominate the simple two-price mechanism.

Why doesn't the second welfare theorem apply in this setting? Any Pareto-efficient allocation is theoretically achievable if agents' initial endowments are appropriately redistributed before trading. However, this arbitrary redistribution of endowments violates individual-rationality and incentive-compatibility constraints since agents would not truthfully reveal their wealth if they knew the designer would redistribute it before trading. Therefore, the mechanism is limited to a subset of allocations in which incentive-compatibility constraints are satisfied so only a constrained efficiency can be achieved.

I first introduce a motivating example with two consumers, Bob and Elon, where Bob has a low wealth and Elon has a high wealth. To simplify the problem, I assume that consumers can only trade once and there is no cost of production. Both agents have a utility function given by: $u(q, w) = qv + \ln(w)$ where q is the number of goods they receive (0 or 1), v is their value for the good, and w is their wealth. Implementing the two-price mechanism in this setting to maximize a utilitarian social welfare function, I find that Elon pays the high price and Bob receives some probability of obtaining the good for free. This outcome is obviously not ex-post Pareto efficient because the mechanism designer can produce the good for free but if Bob does not win the lottery, he does not receive a good. However, in order to enforce incentive-compatibility constraints, Bob cannot simultaneously get the good guaranteed while Elon pays a price greater than 0 so that the revenue can be redistributed. Even though Bob does not always receive the good, total utility increases under this mechanism due to the positive effects of redistribution. This model seeks to illustrate the main intuitions and tradeoffs of the two-price mechanism in a simplified setting. I show that for any utility function of the form $u(q, w) = qv + f(w)$ where $f(w)$ is monotonic and exhibits wealth effects so $f'(w) > 0$ and $f''(w) < 0$, rationing becomes optimal if Bob's marginal utility for money is twice as large as Elon's marginal utility for money where these marginal utilities are not fixed but are a function of their respective wealth.

Next, I investigate the case where there are an infinite number of consumers on a continuum of wealth and focus on the case where wealth is uniformly distributed from $\underline{w}$ to $\overline{w}$. Because of the difficulty of finding an analytical solution, I conduct a numerical analysis of the solution. I find that the wealth ratio $\left(\text{defined as } \frac{\overline{w}}{\underline{w}}\right)$ must exceed around 13 for rationing to be optimal.

I then discuss the policy implications of the model and find that the two-price mechanism could be applied to congestion pricing which fits many of the assumptions in the model. Because the negative externality of congestion features largely in the congestion pricing literature, I modify the Bob and Elon model to account for this externality and find similar results to the initial model. Notably, the introduction of the lottery to probabilistically give away access to the good improves total welfare more than just allocating the good deterministically to consumers.

The remainder of the paper is organized as follows. Section 1.1 reviews the related literature on rationing and mechanism design without quasilinear preferences. Section 2 builds a motivating example of a simple two-consumer case. Section 3 presents the model with a continuum of consumers, Section 3.1 lays out the first steps to analytically solving the model, and Section 3.2 presents the numerical analysis. Section 4 discusses policy implications, specifically in the context of congestion pricing. And Section 5 concludes.

1.1 Related Literature

Weitzman (1977) considered the problem of the equitable distribution of a good quite early on, which arose from the observation that willingness to pay is distinct from someone's valuation of the good, especially for goods that are considered essential. Weitzman concludes that uniform rationing can be better than the price system when there is high variance in wealth but not in consumers' values for the good. This shows the main intuition that price controls can be welfare-enhancing if the social planner prefers redistribution.

There is a small but relevant literature examining non-quasilinear preferences in a mechanism design setting. Kazamura, Mishra, and Serizawa (2020) study an abstract model of mechanism design without quasilinearity. They prove some general properties of dominant strategy incentive-compatible mechanisms in this setting and extend their results to optimal contract design. Baisa (2017) considers the canonical single object auction model without the assumption of quasilinear preferences, finding that it leads to qualitatively different solutions to the original problem. He introduces a probabilistic mechanism and studies its optimality properties in terms of revenue maximization. Garratt and Pycia (2016) examine the bilateral trade model with non-quasilinear preferences and find that efficient trade is possible under certain situations. This result is in contrast to the famous theorem of Myerson and Satterthwaite (1983) of the impossibility of an incentive-compatible, individually rational, budget-balancing, ex-post Pareto efficient mechanism for bilateral trade with quasilinear preferences. Efficient trade becomes possible under non-quasilinearity because wealth effects open a new door for efficiency improvements. Garratt and Pycia introduce a probabilistic mechanism in the form of a lottery contract and prove possibility theorems on efficient trade. Probabilistic mechanisms play a large role in the non-quasilinear setting because it allows the mechanism designer to exploit differences in agents' willingnesses to pay that are independent of their valuations of the good. Similar to the previous literature, this paper utilizes a probabilistic mechanism as well.

Dworczak, Kominers, and Akbarpour (2021) diverged from the canonical model of mechanism design in a very similar fashion to this paper by exploring efficient mechanisms in a buyer-seller marketplace where consumers have varied marginal utilities of money due to wealth inequality. Specifically, they model a market for an indivisible good with many buyers and sellers, each with unit demand or unit supply. Each agent has both a private value for the good and a marginal utility for money, which results in a rate of substitution that completely characterizes each agent's behavior. They find that maximizing total utility is equivalent to maximizing the utilitarian welfare function with an individual's Pareto weight equal to their expected marginal utility for money. The main result of their paper is that in a setting where agents have different marginal utilities for money, the competitive equilibrium is not always optimal and the optimal mechanism may ration the agents in order to redistribute wealth. On the buyer-side, they proved the optimality of the two-price mechanism in the case where agents' rates of substitution are uniformly distributed. This paper extends the model from Dworczak, Kominers, and Akbarpour to a more general case where consumers do not have a constant marginal utility for money but have utility as a function of wealth.

2 The Two Consumer Case: A Motivating Example

Suppose I have free cellphones to give away and there exist two people, Bob and Elon, who both share some value for their first cellphone but don't care about any subsequent cellphones they receive. So, the cellphones are an indivisible good and Bob and Elon have unit demand for them. Elon is wealthier than Bob and has wealth w_H while Bob has wealth w_L. Bob and Elon derive utility only from the number of cellphones they receive and their wealth and they both experience wealth effects, implying diminishing marginal utility in wealth.

Assuming a utilitarian social welfare function, one potential solution is giving both Bob and Elon a free cellphone which would increase total utility by the sum of their values.

However, because Elon is wealthier than Bob, his marginal utility of money is smaller than Bob's marginal utility for money which allows for improvements in total utility if wealth can be distributed. By the lump sum principle, it would be efficient to forcefully take Elon's wealth and give it to Bob but this may be infeasible in the real world if each individual's valuations and wealth are unobservable. So I would therefore like to introduce an incentive-compatible mechanism without an explicit threat by which Elon willingly gives his wealth away because it is in his best interest. This is where the two-price mechanism comes in. Under certain conditions ($w_H > 2w_L$), I show that it is optimal to set two prices: a high price at which Elon purchases the cellphone and a low price at which Bob receives the cellphone with some probability that is less than 1. Because I am unable to price discriminate based on the identity of the consumer, charging Elon for the cellphone means that I must decrease the probability that Bob receives a cellphone so that Elon is not attracted to the low price. So it is the constraint of incentive-compatible redistribution that results in this counter-intuitive result.

This outcome is clearly not ex-post Pareto efficient as I have more cellphones that I could give away for free but I have artificially restricted the supply such that Bob does not always get a cellphone. Because the marginal cost of producing an additional cellphone is zero, the neoclassical approach would seek the competitive equilibrium and set the price to zero. While the notion of the "competitive equilibrium" is not very sensible in this setting with 2 agents and the mechanism designer as the sole supplier, it corresponds to the case where the price is set to the marginal cost of production (which is 0) so that Bob and Elon get the cellphone for free. With a utilitarian social welfare function, I show that when inequality is high enough, the benefit of redistribution outweighs the loss in allocative efficiency so there will be rationing in the market. Next, the model is laid out formally to solve for optimal parameters.

2.1 Solving for Optimal Parameters

Expressing the model more formally, Bob and Elon both value their first cellphone with value v and have 0 value for every subsequent cellphone. So their expected utility functions are defined as $u(q, w) = qv + \ln(w)$ where $q \in [0, 1]$ is the number of cellphones they receive in expectation. An important assumption of this model is that I cannot price discriminate and so a feasible mechanism requires that Bob and Elon reveal their type (wealth) truthfully.

Using the two-price mechanism, I set two prices: the high price, p_H, at which a cellphone can be purchased and the low price, p_L, at which a raffle ticket with some probability, δ, of obtaining the cellphone can be purchased. Setting p_L to 0 is optimal because the only benefit to increasing the low price is the increased redistribution to all consumers which means that the people that are hurt by an increase in the low price must be wealthier than the median consumer. In this case, increasing the low price redistributes some of Bob's wealth to Elon which is clearly suboptimal. Setting $p_L = 0$, I can then choose δ and p_H to maximize total utility after redistribution.

Looking for solutions that actually use the mechanism rather than degenerating into giving two cellphones away for free, I assume that Bob gets the raffle ticket for free since $p_L = 0$ and Elon purchases the cellphone at the high price. So their total utility in expectation is $V = \delta v + \ln(w_L - p_L + r) + v + \ln(w_H - p_H + r)$ where r is the lump-sum redistribution per consumer which is the total revenue divided by the number of consumers. The total revenue for this mechanism is p_H so $r = \frac{p_H}{2}$ and the objective function can be rewritten as

$$V = \delta v + \ln\left(w_L + \frac{p_H}{2}\right) + v + \ln\left(w_H - \frac{p_H}{2}\right)$$

"Rationing" is defined as the case where Bob does not receive the cellphone with 100% probability. In this model, a rationing solution means setting $\delta < 1$ and $p_H > 0$. If rationing is optimal, then I would want to maximize redistribution given some δ, which means Elon's incentive compatibility constraint should bind. So the parameters should be set such that

Elon is indifferent between the high price and the free raffle ticket. This is similar to the approach used in the screening literature where the low type's participation constraint binds and the high type's incentive-compatibility constraint binds. In this case, Bob's participation constraint does not bind because he always gets positive benefits from the mechanism. Elon's incentive-compatibility constraint is given by $\delta v + \ln(w_H + r_L) = v + \ln(w_H - p_H + r_H)$ where the r_L is the redistribution per consumer if Elon chooses the raffle ticket and r_H is the redistribution per consumer if Elon chooses the high price. The LHS of the expression represents Elon's utility if he gets the free raffle ticket while the RHS represents Elon's utility if he buys the cellphone at the high price. In this case, with Bob getting the raffle ticket at $p_L = 0$, Elon choosing the raffle ticket makes $r_L = 0$ while choosing the high price makes $r_H = \frac{p_H}{2}$. So, Elon's incentive-compatibility constraint can be rewritten to be:

$$\delta v + \ln(w_H) = v + \ln\left(w_H - \frac{p_H}{2}\right)$$

Solving, I find that Elon is indifferent between the high price and the raffle ticket if

$$w_H = \frac{e^{v(1-\delta)} p_H}{2\left(e^{v(1-\delta)} - 1\right)} \tag{1}$$

So I aim to maximize total utility subject to Elon's incentive-compatibility constraint:

$$\max_{p_H, \delta} \delta v + \ln\left(w_L + \frac{p_H}{2}\right) + v + \ln\left(w_H - \frac{p_H}{2}\right) \; s.t. \; w_H = \frac{e^{v(1-\delta)} p_H}{2\left(e^{v(1-\delta)} - 1\right)} \tag{2}$$

Solving explicitly, I get the unrestricted optimal parameters for this mechanism:

$$p_H^* = \frac{2}{3}(w_H - 2w_L) \tag{3}$$

$$\delta^* = 1 + \frac{1}{v} \ln\left(\frac{2(w_H + w_L)}{3w_H}\right) \tag{4}$$

As a probability, δ should be restricted to range from 0 to 1. So it makes more sense to describe δ^* as a piecewise function that is capped by 0 and 1.

$$\delta^*(w_H, w_L, v) = \begin{cases} 0 & \text{if } \frac{w_H}{w_L}(3e^{-v} - 2) > 2 \\ 1 + \frac{1}{v}\ln\left(\frac{2(w_H+w_L)}{3w_H}\right) & \text{if } \frac{w_H}{w_L}(3e^{-v} - 2) < 2 \text{ and } \frac{w_H}{w_L} \geq 2 \\ 1 & \text{if } \frac{w_H}{w_L} < 2 \end{cases} \tag{5}$$

The piecewise conditions for δ^* are determined by solving equation (4) when $\delta^* = 0$ or $\delta^* = 1$. Because δ^* is now being restricted, I similarly have to modify p_H^* in these regions.

$$p_H^*(w_H, w_L, v) = \begin{cases} 2w_H\left(1 - \frac{1}{e^v}\right) & \text{if } \frac{w_H}{w_L}(3e^{-v} - 2) > 2 \\ \frac{2}{3}(w_H - 2w_L) & \text{if } \frac{w_H}{w_L}(3e^{-v} - 2) < 2 \text{ and } \frac{w_H}{w_L} \geq 2 \\ 0 & \text{if } \frac{w_H}{w_L} < 2 \end{cases} \tag{6}$$

When $\frac{w_H}{w_L} < 2$, the optimal solution is to give a free cellphone to both Bob and Elon so the high price is irrelevant.

I check the second order conditions in the appendix and show that this problem is globally concave, confirming this solution as the global maximum.

2.2 Interpretation of Optimal Parameters

2.2.1 Raffle Probability vs the Wealth Ratio

Note that δ^* is only dependent on the wealth ratio $\frac{w_H}{w_L}$ and not the values themselves. Figure 1 illustrates δ^* as a function of the wealth ratio.

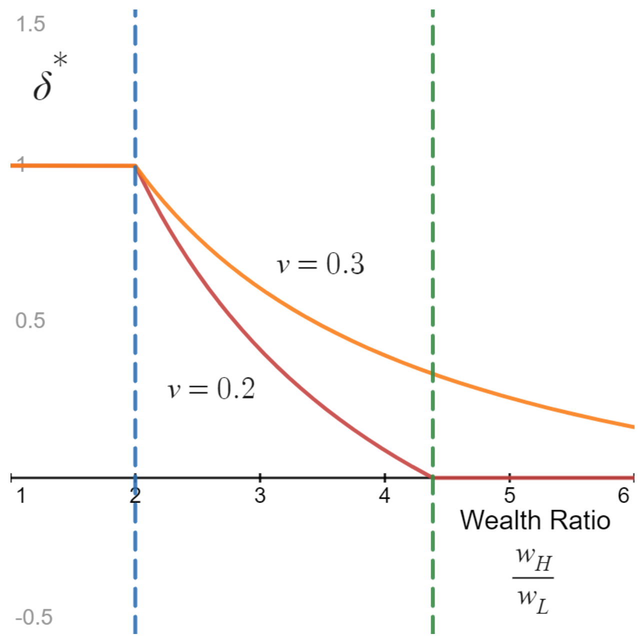


Figure 1: This graph shows the optimal value of delta versus the wealth ratio between Elon and Bob where $x = \frac{w_H}{w_L}$. The red line shows $v = 0.2$ and the orange line shows $v = 0.3$.

δ^* is decreasing in the wealth ratio. This makes sense as the mechanism increasingly favors redistribution through the high price rather than allowing Bob some probability to obtain the cellphone through the raffle ticket. Due to the incentive compatibility constraint on Elon, making him indifferent between the raffle ticket and the high price, the only way to increase redistribution to Bob must be to decrease δ and make the raffle ticket less attractive. Notice that $\delta^*(v = 0.3) \geq \delta^*(v = 0.2)\ \forall \frac{w_H}{w_L}$. All else equal, if the consumers value the cellphone more, then it is given away more freely, allowing Bob a higher probability to win the cellphone.

Define $\left(\frac{w_H}{w_L}\right)^*$ as the minimum wealth ratio at which rationing becomes optimal. If the wealth ratio between Bob and Elon is greater than $\left(\frac{w_H}{w_L}\right)^* = 2$, then there exists a rationing solution that provides strictly greater total utility than the unique incentive-compatible ex-post Pareto efficient allocation. The unique incentive-compatible ex-post Pareto efficient allocation is the case where Bob and Elon both get the cellphone for free. When there is rationing, the ex-post allocation is not Pareto efficient since Bob does not receive the cellphone with probability $1 - \delta$ when there is unused leftover supply. The next section shows why rationing becomes optimal when the wealth ratio exceeds 2.

2.2.2 Conditions for Rationing

This result on the conditions for rationing applies more generally to any utility function of the form $u(q, w) = qv + f(w)$ where $f(w)$ captures the notion of wealth effects by exhibiting monotonicity and diminishing marginal utility: $f'(w) > 0$ and $f''(w) < 0$.

Proposition 1. *For any utility function of the form $u(q, w) = qv + f(w)$ where $f(w)$ is monotonic and strictly concave, rationing becomes optimal when $MU_B > 2MU_E$. A rationing solution is defined as an optimal $\delta < 1$ and $p_H > 0$.*

Proof. Assume that $p_H = 0$ and $\delta = 1$ and I am trying to decide whether to increase p_H by ϵ and induce rationing on the margin. If I increase p_H, then there must be a corresponding decrease in δ so that Elon's incentive-compatibility constraint still binds so I want to find out how large this decrease is. His IC constraint in this more general setting is $u\left(1, w_H - \frac{p_H}{2}\right) = \delta u(1, w_H) + (1 - \delta)u(0, w_H)$. Plugging in the specific form of the utility function, I get $v + f\left(w_H - \frac{p_H}{2}\right) = \delta v + f(w_H)$ where the LHS represents Elon's utility from purchasing the good at the high price and the RHS represents Elon's utility from taking a free δ probability to win the good. The IC constraint still holds if I take the total derivative of both sides with respect to p_H.

$$\frac{d}{dp_H}\left(v + f\left(w_H - \frac{p_H}{2}\right)\right) = \frac{d}{dp_H}(\delta v + f(w_H))$$

By the chain rule, $\frac{d}{dp_H} = \frac{d}{dw}\frac{dw}{dp_H}$ and $\frac{dw}{dp_H} = 1$ so

$$-\frac{1}{2}\frac{df\left(w_H - \frac{p_H}{2}\right)}{dw} = \frac{d\delta}{dp_H}v$$

I am evaluating this when $p_H = 0$ so

$$-\frac{1}{2}\frac{df(w_H)}{dw} = \frac{d\delta}{dp_H}v$$

Intuitively, this should make sense as a small increase in p_H should decrease Elon's utility by something that is proportional to his marginal utility for money and to keep him indifferent between the high price and the low price, δ must also decrease by something that is proportional to his value for the good. The change in δ when I increase p_H by ϵ is then $\Delta\delta = \frac{d\delta}{dp_H}\Delta p_H = \frac{d\delta}{dp_H}\epsilon$. So the high price increases by ϵ and δ decreases by $-\Delta\delta$. The marginal benefit of increasing the high price is the redistribution that Elon and Bob receive. In this case, $r = \frac{\epsilon}{2}$ since Elon purchases the good at $p_H = \epsilon$.

$$MB = \frac{\epsilon}{2}MU_B + \frac{\epsilon}{2}MU_E$$

where MU_B is Bob's marginal utility of money and MU_E is Elon's marginal utility of money. The marginal cost of increasing the high price is the loss of utility to Elon who is paying ϵ more than before. Also, δ decreases so Bob loses some probability of obtaining the good which decreases his utility by $-v\Delta\delta$ since $\Delta\delta$ is negative. This term represents the loss in allocative efficiency in order to enforce Elon's incentive-compatibility constraint.

$$MC = \epsilon MU_E - v\Delta\delta$$

Using the expression for $\Delta\delta$ from earlier, I get $v\Delta\delta = -\frac{\epsilon}{2}\frac{df(w_H)}{dw}$ so $MC = \epsilon MU_E + \frac{\epsilon}{2}\frac{df(w_H)}{dw}$. MU_E is defined as $\frac{df(w_H)}{dw}$ so $MC = \frac{3\epsilon}{2}MU_E$. Equating the marginal benefit and marginal cost, I get $\frac{\epsilon}{2}MU_B + \frac{\epsilon}{2}MU_E = \frac{3\epsilon}{2}MU_E$ so marginal benefit equals marginal cost when $\frac{1}{2}MU_B = MU_E$ or when Bob's marginal utility for money is twice as large as Elon's. Because $f''(w) < 0$, $f'(w_L) > f'(w_H)$ if $w_L < w_H$ so $MU_B > MU_E$. If $MU_B > 2MU_E$, MB exceeds MC and so rationing is optimal relative to giving the good away for free to both Bob and Elon. □

For log utility in wealth, $MU_E = \frac{1}{w_H}$ and $MU_B = \frac{1}{w_L}$ so rationing becomes optimal when $w_H = 2w_L$. So if $w_H < 2w_L$, the marginal cost exceeds the marginal benefit, and if $w_H > 2w_L$, the marginal benefit exceeds the marginal cost. This condition notably does not depend on Bob and Elon's value for the cellphone. Since this problem is globally concave, if the marginal benefit exceeds the marginal cost at $p_H = 0$, the optimal p_H is greater than 0 and an interior solution exists.

This derivation also illustrates why the unrestricted p_H^* in equation (3) is constant for all v. The decision to ration on the margin is independent of v, so I would want to keep increasing the high price until $MU_B = 2MU_E$ or in the log case, $w_H = 2w_L$. And that is exactly what the mechanism does. Recall the unrestricted high price $p_H^* = \frac{2}{3}(w_H - 2w_L)$. Elon's wealth after trading is $w_H - \frac{p_H^*}{2}$ and Bob's wealth after trading is $w_H + \frac{p_H^*}{2}$ since $r = \frac{p_H^*}{2}$. Simplifying, Elon's final wealth is $\frac{2}{3}(w_H + w_L)$ and Bob's final wealth is $\frac{1}{3}(w_H + w_L)$, so Elon's final wealth will be twice as large as Bob's final wealth such that additional rationing is no longer desirable. δ^* is then appropriately adjusted to hold Elon's incentive-compatibility constraint.

2.2.3 The Optimal Parameters vs the Value

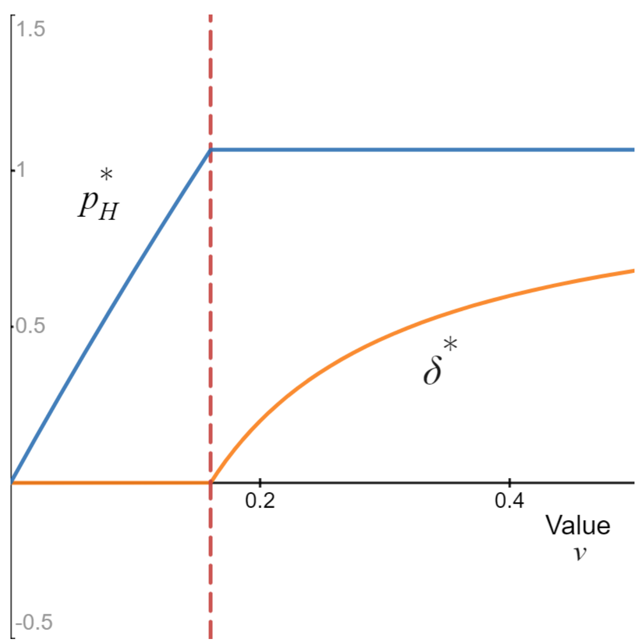


Figure 2: This graph shows δ^* and p_H^* versus Bob's and Elon's value for the cellphone. w_H was set to 3.6 and w_L was set to 1, so $\frac{w_H}{w_L} = 3.6$. Any wealth ratio less than 2 is nonsensical since rationing is not optimal.

Proposition 2. *For a fixed wealth ratio greater than* $\left(\frac{w_H}{w_L}\right)^* = 2$*, there exists a cutoff value* $v^* = \ln\left(\frac{3\frac{w_H}{w_L}}{2\left(\frac{w_H}{w_L}+1\right)}\right)$ *where the behavior of the optimal parameters shifts. For* $v \leq v^*$*,* p_H^* *is increasing in* v *and* δ^* *is set to 0. For* $v > v^*$*,* p_H^* *is fixed at* $\frac{2}{3}(w_H - 2w_L)$ *and* δ^* *asymptotically approaches 1 as* $v \to \infty$*.*

In figure 2, $\delta^* = 0$ when $v \leq v^* = \ln\left(\frac{3w_H}{2(w_H+w_L)}\right) \approx 0.16$. Fixing the wealth ratio, δ^* is increasing in Bob's and Elon's value for the cellphone and asymptotically approaches 1. Intuitively, Bob should get a higher probability to win the cellphone as his value for the cellphone increases. The dotted red line marks the cutoff value v^* at which there is a kink in the behavior of the optimal parameters. This kink happens because δ must be greater than or equal to 0. Notice that in the unrestricted optimal parameters in (3) and (4), p_H^* is constant with respect to v and δ^* can be less than 0 and greater than 1. So the unrestricted optimal parameters would set p_H^* constant $\forall v$ and a negative δ^* for $v < v^*$. However, δ^* cannot be less than 0, so this creates a kink in the optimal parameters. p_H^* must then decrease before the cutoff value as Elon values the cellphone less so his willingness to pay decreases.

2.3 Introducing Scarcity

Say the good is not plentiful so the mechanism designer faces scarcity, having only x units of the good to distribute and introducing a supply constraint in the market. Although I have specified that the good is indivisible, allowing x to be a real number better illustrates how scarcity affects the solutions to the problem. So, imagine that instead of one Bob and Elon there are infinitely many Bobs and Elons (equally many) and specify x as a fraction of the total number of agents that can consume the good so $x \in [0, 1]$. More formally, there is a unit mass of consumers with measure 0.5 Bobs and measure 0.5 Elons. The supply constraint is that I cannot give the good to more than x fraction of the consumers.

When there are infinitely many agents, this slightly changes Elon's incentive-compatibility constraint since his decision to take the low price does not change the revenue of the mechanism. When there are just two agents, Elon's decision to take the low price means he gets $r = 0$ but with infinitely many consumers, he would still get $r = \frac{p_H}{2}$ since all the other Elons are assumed to stay at the high price. This changes the solution slightly, but not in any meaningful way. Introducing scarcity in this problem results in two different types of solutions depending on x.

2.3.1 Case 1: $0 \leq x < 0.5$

Without scarcity, it has always been assumed that Elon trades at the high price and Bob gets the raffle ticket for free. However, if $0 \leq x < 0.5$, I am unable to offer guaranteed trade at the high price since this requires $x \geq 0.5$ so I can only offer the raffle ticket at the low price p_L. All Elons will act in the same way so if I offer a high price that is attractive to any Elon, 50% of the agents (all the Elons) will try to purchase the good at the high price. But $x < 0.5$ so I do not have the supply necessary to give all the Elons the good. So imagine that the high price is prohibitively high such that no agents trade there or that it is not an option anymore. Because Bob and Elon both derive the same marginal utility from the good v, total utility increases the same amount whether the good goes to Bob, Elon, or is split among them. Elon's willingness to pay for the good is higher than Bob so the mechanism designer can induce welfare-enhancing redistribution from Elon to Bob by setting $p_L > 0$ and giving all x of the good to Elon. Since the Elons constitute 50% of the agents, I can offer each of them $\delta = 2x$ probability to win the good in order to distribute all x of the good. So the optimal parameters are given by $\delta^* = 2x$ and $p_L^* = \frac{2w_H(e^{2xv}-1)}{e^{2xv}+1}$ where p_L^* is just the low price at which Elon is indifferent between not trading and the low price given that $\delta = 2x$. Note that there is no leftover supply in this case as the good is always entirely consumed by Elon.

2.3.2 Case 2: $0.5 \leq x \leq 1$

If $0.5 \leq x \leq 1$, I am able to reintroduce guaranteed trade at the high price. Holding the amount of the good constant, I should choose the allocation that maximizes the redistribution from Elon to Bob so I allow all Elons to trade at the high price. Similar to when rationing is optimal without scarcity, Elon trades at the high price and Bob gets a raffle ticket where $p_L = 0$. Given $0.5 \leq x < 1$, there is enough supply for all Elons to get the good at the high price and for all Bobs to get $\delta = 2(x - 0.5)$ probability to win the good. Given the initial parameters of the problem (w_H, w_L, v), if the original δ^* in equation (5) is less than $2(x-0.5)$, then the optimal parameters with scarcity are identical to the optimal parameters without scarcity and there is leftover supply. However, if $\delta^* > 2(x - 0.5)$, then the supply constraint binds so I would set $\delta = 2(x - 0.5)$, giving Bob the remainder of the good after Elon trades at the high price. The high price is set such that Elon is indifferent between the low price and the high price given that value of δ.

Introducing scarcity does not fundamentally alter how the solutions look. The solutions would look nearly identical if there were two agents and a scarce divisible good with x ranging from 0 to 1. The supply constraint obviously limits the cases in which there is unused leftover supply, otherwise, the main themes of the problem remain the same.

2.4 Discussion

The main result from this toy example is that providing this costless good to both consumers for free is not ex-ante optimal if wealth inequality is severe enough. A sufficiently large difference in the consumers' marginal utilities for money allows the gains from redistribution to offset the loss in allocative efficiency. With $1 - \delta$ probability, Bob does not receive the cellphone, resulting in this counterintuitive ex-post Pareto inefficient outcome. The mechanism designer could increase total utility by v if they gave the cellphone to Bob even if he did not win the lottery. However, once this is done, the mechanism's incentive compatibility breaks down and Elon would never purchase the cellphone at the high price knowing that he would get the cellphone even if he takes the free raffle ticket. So the mechanism designer must be able to credibly threaten to throw the cellphone away if the raffle ticket lottery is lost. In this setting, it seems likely that the two-price mechanism is optimal among all incentive-compatible mechanisms but this has not been proven.

3 Model

I study a marketplace of consumers with a mechanism designer that has some supply of an indivisible good. All consumers have identical utility functions which are additively separable and are broken up into the utility they receive from the good and from their wealth:

$$u(v, w_i) = q(v, w_i)v + \ln(w_i) \tag{7}$$

v is a consumer's value for the good which I assume is the same for all consumers, w_i is the consumer's private type of their wealth, and $q(v, w_i)$ is the quantity of the good that the agent receives. I proceed by assuming that consumers exhibit log utility in wealth.

Like in the example with Bob and Elon, the consumers participate in a two-price mechanism that is determined by three parameters, the high price, p_H, the low price, p_L, and the probability of trading at the low price, δ, subject to market clearing constraints, incentive compatibility constraints, and individual rationality constraints. Consumers can either trade at the high price at which trade is guaranteed, trade at the low price at which the buyer has some probability of obtaining the good, or not trade at all. Up until this point, I have assumed that $p_L = 0$, but here I discuss the more general two-price mechanism where p_L is not necessarily 0. To simplify this problem, I assume that all consumers will only be able to trade once. After each consumer makes their choice, the mechanism design collects the profits from the mechanism and evenly redistributes them back to the buyers as a lump-sum payment.

I start by assuming that the mechanism designer does not have a supply constraint. The mechanism designer chooses values of p_H, p_L, and δ such that their objective function is maximized and this may result in leftover supply for the mechanism designer. One easy way to distribute the goods is to freely provide one unit of the good to each consumer; however, this is not always optimal and the redistributive effect of setting positive prices may outweigh the cost of having some amount of supply left unconsumed. In this case, I re-derive the counterintuitive result from the example with Bob and Elon that an ex-post Pareto inefficient outcome can be optimal for a utilitarian social welfare function in order to induce redistribution.

3.1 Analytical Results

3.1.1 Setting up the Maximization Problem

Let there be an infinite number of consumers distributed from $\underline{w}$ to $\overline{w}$ with some density function $\rho(w)$. I maximize a utilitarian social welfare function which is equivalent to maximizing the average utility[1] of a consumer which is

$$\overline{U} = \int_{\underline{w}}^{\overline{w}} u(v, w)\rho(w)\, dw \tag{8}$$

Assuming $p_L < p_H$, define w_L^* to be the "cutoff" wealth at which consumers are indifferent to not purchasing the good and purchasing the good at the low price with some probability to receive the good δ. Similarly, define w_H^* to be the cutoff wealth at which consumers are indifferent to purchasing the good at the low price and purchasing the good at the high price. These two points are salient because all consumers with wealth less than w_L^* do not purchase the good, all consumers with wealth $w_L^* < w < w_H^*$ purchase the good at the low price, and all consumers with wealth greater than w_H^* purchase the good at the high price. Assuming that $\underline{w} < w_L^* < w_H^* < \overline{w}$, these two cutoff points split the consumers into three distinct groups that decide the outcome of the mechanism, shown in Figure 3. There can

1. $u(v, w)$ is technically the consumer's expected utility since they face a lottery and is treated as such throughout the paper.

be instances in which w_L^* or w_H^* are below $\underline{w}$, in which case the consumers are split into two groups where one group purchases the good at the low price and the other at the high price or that all consumers buy the good at the high price.

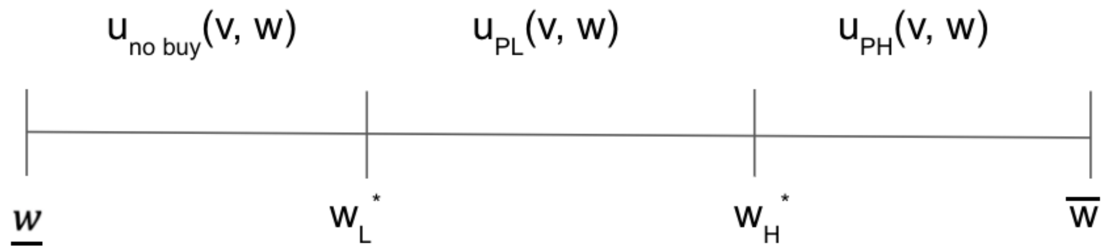


Figure 3: This figure shows the continuum of consumer wealths when $\underline{w} < w_L^* < w_H^* < \overline{w}$.

After trading concludes, all the profits received in the mechanism are redistributed evenly as a lump-sum payment to all consumers. The redistribution per buyer is calculated by looking at what proportion of consumers chose to trade at the low price or high price, weighted by their density. The redistribution per buyer is given by

$$r = p_L \int_{w_L^*}^{w_H^*} \rho(w)\, dw + p_H \int_{w_H^*}^{\overline{w}} \rho(w)\, dw \tag{9}$$

where the fraction of consumers at the low price is the first term excluding p_L and the fraction of consumers at the high price is the second term excluding p_H.

The three groups of consumers have post-trading utility functions that are given by equation (7),

$$u_{no\,buy}(v, w) = \ln(w + r) \tag{10}$$

for the group who does not buy the good where r is the redistribution per buyer in (9),

$$u_{p_L}(v, w) = \delta v + \ln(w - p_L + r) \tag{11}$$

for the group who buys the good at the low price, and

$$u_{p_H}(v, w) = v + \ln(w - p_H + r) \tag{12}$$

for the group who buys the good at the high price.

At w_L^*, there is an indifference condition for the consumer who is indifferent to not buying the good and buying it at the low price, so plugging in w_L^* into (10) and (11), I get:

$$\ln(w_L^* + r) = \delta v + \ln(w_L^* - p_L + r)$$

Because there an infinite number of consumers, a marginal consumer's switching from the low price to not purchasing the good is negligible compared to the average redistribution per buyer so r is the same on both sides of the equation. Simplifying further, I get:

$$w_L^* = \frac{p_L e^{\delta v}}{e^{\delta v} - 1} - r \tag{13}$$

Similarly, I go through the same procedure at w_H^* for the consumer who is indifferent to buying some fraction of the good (in expectation) at the low price and buying the good guaranteed at the high price. The indifference condition in this case is:

$$\delta v + \ln(w_H^* - p_L + r) = v + \ln(w_H^* - p_H + r) \tag{14}$$

I obtain a corresponding expression for w_H^*:

$$w_H^* = \frac{e^{v(1-\delta)} p_H - p_L}{e^{v(1-\delta)} - 1} - r \tag{15}$$

Note that in both (13) and (15), there is the r term on the RHS which is dependent on both w_L^* and w_H^*, making this a system of two equations. Ignoring the r term for now, there is a fair amount of economic intuition for these two expressions for w_L^* and w_H^* that reflect the tradeoffs that consumers face when deciding between these three options. The expression for w_L^* breaks down when δ equals 0, making the denominator 0. This makes sense as w_L^* should go off towards infinity as the probability of trading becomes very low as it becomes an increasingly unattractive trade. Also, $w_L^* \leq 0$ when p_L is 0 as trading the good with some positive probability for free is strictly preferred to not trading at all wealth levels. In general, w_L^* is increasing in p_L as higher prices make the trade unattractive to those with lower wealth. Similarly, the expression for w_H^* implies that δ cannot equal 1 as the low price would be strictly preferred to the high price. w_H^* is also increasing in p_H and decreasing in p_L since increasing p_H limits access to the good while increasing p_L makes the low price less attractive to consumers on the margin, making them switch to the high price.

So if $\underline{w} \leq w_L^* \leq w_H^* \leq \overline{w}$, the mechanism designer's objective function can be represented as

$$V = \int_{\underline{w}}^{w_L^*} \ln(w+r)\rho(w)\,dw + \int_{w_L^*}^{w_H^*} (\delta v + \ln(w-p_L+r))\rho(w)\,dw + \int_{w_H^*}^{\overline{w}} (v+\ln(w-p_H+r))\rho(w)\,dw \tag{16}$$

where r is the redistribution per buyer in (9). The mechanism designer's goal is to maximize this quantity with respect to p_L, p_H, and δ subject to the constraints that $p_L \geq 0$, $p_H \geq 0$, $p_H \geq p_L$, $0 \leq \delta \leq 1$, and $\underline{w} \leq w_L^* \leq w_H^* \leq \overline{w}$.

3.1.2 Simplifying Assumptions

I first solve assuming that consumers are uniformly distributed from $\underline{w}$ to $\overline{w}$. This means that the density of consumers $\rho(w) = \frac{1}{\overline{w}-\underline{w}}$ from $\underline{w}$ to $\overline{w}$, and 0 everywhere else. Therefore, the objective function can be reformulated using (8) to be

$$\overline{U} = \frac{1}{\overline{w}-\underline{w}} \int_{\underline{w}}^{\overline{w}} u(v,w)dw$$

Maximizing $\overline{U}$ is identical to maximizing $\int_{\underline{w}}^{\overline{w}} u(v,w)dw$ because $\frac{1}{\overline{w}-\underline{w}}$ is a constant so I remove it for simplicity. So more explicitly, the new objective function is identical to (16) but without the $\rho(w)$ term.

$$V = \int_{\underline{w}}^{w_L^*} \ln(w+r)\, dw + \int_{w_L^*}^{w_H^*} \delta v + \ln(w - p_L + r)\, dw + \int_{w_H^*}^{\overline{w}} v + \ln(w - p_H + r)\, dw \tag{17}$$

With uniformly distributed consumers, the redistribution per buyer also simplifies from (9) when I plug in $\rho(w) = \frac{1}{\overline{w}-\underline{w}}$:

$$r = \frac{w_H^* - w_L^*}{\overline{w} - \underline{w}} p_L + \frac{\overline{w} - w_H^*}{\overline{w} - \underline{w}} p_H \tag{18}$$

With some evidence from brute force simulated optimization, it seems that the optimal low price could be 0 and this provides a nice way to simplify this problem. If I assume that $p_L = 0$, then $w_L^* = -r$ from (13) which means that all consumers with positive wealth choose the low price or the high price. So the pricing scheme now segments the consumers into two groups instead of three. This simplifies the objection function from (17) to a new maximization problem, still assuming that $\underline{w} \leq w_H^* \leq \overline{w}$:

$$\max_{p_H, \delta} \int_{\underline{w}}^{w_H^*} \delta v + \ln(w+r)\, dw + \int_{w_H^*}^{\overline{w}} v + \ln(w - p_H + r)\, dw \tag{19}$$

From (18), I can remove the term with p_L and get

$$r = \frac{\overline{w} - w_H^*}{\overline{w} - \underline{w}} p_H \tag{20}$$

Plugging this into the expression for w_H^* in (15), I get an explicit expression for w_H^*:

$$w_H^* = \frac{p_H \left(\overline{w} - e^{v(1-\delta)} \underline{w}\right)}{\left(e^{v(1-\delta)} - 1\right)\left(\overline{w} - \underline{w} - p_H\right)} \tag{21}$$

This form of the objective function and redistribution are nonsensical if $\underline{w} \leq w_H^* \leq \overline{w}$ does not hold. This would mean that all the consumers either buy the good at the high price or receive δ probability to win the good (at the low price of 0). For example, if w_H^*

can exceed $\overline{w}$, then the amount of redistribution could be negative according to (20), but because it no longer accurately describes consumer behavior at this point. If $w_H^* > \overline{w}$, then all consumers trade at the low price and $r = 0$ so w_H^* is effectively equal to $\overline{w}$. Because of this, it is more accurate to state w_H^* as a piecewise function instead:

$$w_H^*(p_H, \delta) = \begin{cases} \underline{w} & \text{if } \frac{p_H\left(\overline{w}-e^{v(1-\delta)}\underline{w}\right)}{\left(e^{v(1-\delta)}-1\right)(\overline{w}-\underline{w}-p_H)} < \underline{w} \\ \frac{p_H\left(\overline{w}-e^{v(1-\delta)}\underline{w}\right)}{\left(e^{v(1-\delta)}-1\right)(\overline{w}-\underline{w}-p_H)} & \text{if } \underline{w} \leq \frac{p_H\left(\overline{w}-e^{v(1-\delta)}\underline{w}\right)}{\left(e^{v(1-\delta)}-1\right)(\overline{w}-\underline{w}-p_H)} \leq \overline{w} \\ \overline{w} & \text{if } \frac{p_H\left(\overline{w}-e^{v(1-\delta)}\underline{w}\right)}{\left(e^{v(1-\delta)}-1\right)(\overline{w}-\underline{w}-p_H)} > \overline{w} \end{cases} \tag{22}$$

3.1.3 First Order Conditions

FOC: The High Price

With the simplified maximization problem in (19), I take the first order conditions, starting with p_H: $\frac{\partial V}{\partial p_H} = \frac{\partial r}{\partial p_H}(\ln(w_H^* + r) - \ln(\underline{w} + r) + \ln(\overline{w} - p_H + r) - \ln(w_H^* - p_H + r)) - (\ln(\overline{w} - p_H + r) - \ln(w_H^* - p_H + r)) + \frac{\partial w_H^*}{\partial p_H}(\delta v + \ln(w_H^* + r) - (v + \ln(w_H^* - p_H + r))) = 0$

Using the indifference condition at w_H^* from (14), I know that $\delta v + \ln(w_H^* + r) = v + \ln(w_H^* - p_H + r)$ so I can cancel this term in $\frac{\partial V}{\partial p_H}$. Simplifying, I get:

$$\frac{\partial r}{\partial p_H}(\ln(w_H^* + r) - \ln(\underline{w} + r) + \ln(\overline{w} - p_H + r) - \ln(w_H^* - p_H + r)) = \ln(\overline{w}-p_H+r)-\ln(w_H^*-p_H+r) \tag{23}$$

Using (21), I get $\frac{\partial w_H^*}{\partial p_H} = \frac{(\overline{w}-\underline{w})\left(\overline{w}-e^{v(1-\delta)}\underline{w}\right)}{\left(e^{v(1-\delta)}-1\right)(\overline{w}-\underline{w}-p_H)^2}$ so $\frac{\partial w_H^*}{\partial p_H} > 0$ if $\delta < 1$ and $\frac{\overline{w}}{\underline{w}} > e^{v(1-\delta)}$. $\frac{\partial w_H^*}{\partial p_H}$ should be positive since increasing the high price should increase w_H^*, the wealth of that marginal consumer who is indifferent to the high price and the low price, since the high price has become less attractive.

And using (20), I get $\frac{\partial r}{\partial p_H} = \frac{1}{\overline{w}-\underline{w}}\left(-\frac{\partial w_H^*}{\partial p_H}p_H + \overline{w} - w_H^*\right)$ so $\frac{\partial r}{\partial p_H} > 0$ if $\overline{w} - w_H^* > \frac{\partial w_H^*}{\partial p_H}p_H$. The effect of increasing the high price on the amount of redistribution is unclear since increasing the high price causes the consumers currently paying the high price to pay more but also some consumers now switch from paying the high price to paying nothing. This tradeoff is akin to the one demonstrated in the Laffer curve that relates government revenue to the tax rate.

Looking at (23) more carefully, to increase total utility by increasing p_H, the change in the amount of redistribution in response to an increase in the high price, $\frac{\partial r}{\partial p_H}$, must be positive to offset the utility loss for the high paying consumers since they have to pay a higher price. The marginal benefit of increasing the high price and having more redistribution is shown by the LHS of (23), while the marginal cost to the high paying consumers is shown by the RHS. Notably, $\frac{\partial r}{\partial p_H}$ must be less than 1 since p_H is increasing for some consumers but the redistribution amount is paid to all consumers. So increasing the high price is most effective when increasing redistribution to the poorest consumers outweighs the cost to allocative efficiency and the utility loss incurred by the consumers paying the high price.

FOC: Delta

Now I take the first order condition with respect to δ: $\frac{\partial V}{\partial \delta} = \frac{\partial r}{\partial \delta}(\ln(w_H^* + r) - \ln(\underline{w} + r) + \ln(\overline{w} - p_H + r) - \ln(w_H^* - p_H + r)) + v(w_H^* - \underline{w}) + \frac{\partial w_H^*}{\partial \delta}(\delta v + \ln(w_H^* + r) - (v + \ln(w_H^* - p_H + r))) = 0$ Similarly as above, the indifference condition for w_H^* implies that the last term is zero. Simplifying, I get:

$$v(w_H^* - \underline{w}) = -\frac{\partial r}{\partial \delta}(\ln(w_H^* + r) - \ln(\underline{w} + r) + \ln(\overline{w} - p_H + r) - \ln(w_H^* - p_H + r)) \qquad (24)$$

Using (21), I get $\frac{\partial w_H^*}{\partial \delta} = \frac{e^{v(1-\delta)} p_H (\overline{w} - \underline{w})}{\left(e^{v(1-\delta)} - 1\right)^2 (\overline{w} - \underline{w} - p_H)}$so $\frac{\partial w_H^*}{\partial \delta} > 0$ if $\overline{w} > \underline{w} + p_H$. Intuitively, increasing δ increases w_H^* since the low price becomes more attractive which increases the wealth of that marginal consumer.

And using (20), I get $\frac{\partial r}{\partial \delta} = -\frac{\partial w_H^*}{\partial \delta} \frac{p_H}{\overline{w} - \underline{w}}$. Because $\frac{\partial w_H^*}{\partial p_H}$ is usually positive, this makes $\frac{\partial r}{\partial \delta}$ negative. This makes sense as increasing δ makes the low price more attractive which should decrease the number of consumers paying the high price which decreases the revenue of the mechanism.

Looking at (24) more carefully, the LHS of this expression represents the marginal benefit of increasing δ as more consumers are able to receive more of the good in expectation while the RHS represents the marginal cost as increasing delta usually decreases the amount of redistribution as consumers switch from the high price to the low price.

Solving for the optimal parameters analytically turns out to be very difficult because this problem is not globally concave. Additionally, I cannot solve for instances where rationing is optimal at the margin. If I set the parameters such that $w_H* = \underline{w}$ so everyone is purchasing the good at the high price and I am deciding whether or not to increase p_H or δ by ϵ to induce marginal rationing by moving w_H^* into the interior of the interval $(\underline{w}, \overline{w})$, I can take the first derivatives of the value function with respect to p_H and δ. However, it turns out that $\frac{\partial V}{\partial p_H} \leq 0$ and $\frac{\partial V}{\partial \delta} \leq 0$ when $w_H^* = \underline{w}$. So if there is no rationing, any additional marginal rationing is always suboptimal; however, there do exist interior optima since $\frac{\partial V}{\partial p_H}$ and $\frac{\partial V}{\partial \delta}$ eventually become positive and offset the initial losses to total utility. For this reason, I turn to numerical optimization techniques next.

3.2 Numerical Analysis

Because the optimal parameters cannot be solved for explicitly using the first order conditions, I turn to numerical maximization techniques to visualize the solution space. I assign numerical values to the exogenous parameters, $\underline{w}$, $\overline{w}$, and v, and then explore the behavior of the maximized values of the choice parameters, p_H and δ. Figure 4 shows a contour plot of the objective function with respect to p_H and δ.

For this section, I take w_H^* to take the piecewise form seen in (22) so that the solutions are intelligible. So in figure 4, the blank section in the top right of the graph marks where $w_H^* = \overline{w} = 10$. Recall that w_H^* is the consumer with wealth such that they are indifferent between the low price and high price. Similarly, the blank section on the left side of the graph marks where $w_H^* = \underline{w} = 0.1$. Interior solutions I am searching for must obey the condition that $\underline{w} < w_H^* < \overline{w}$. If this condition is not satisfied, then all consumers either

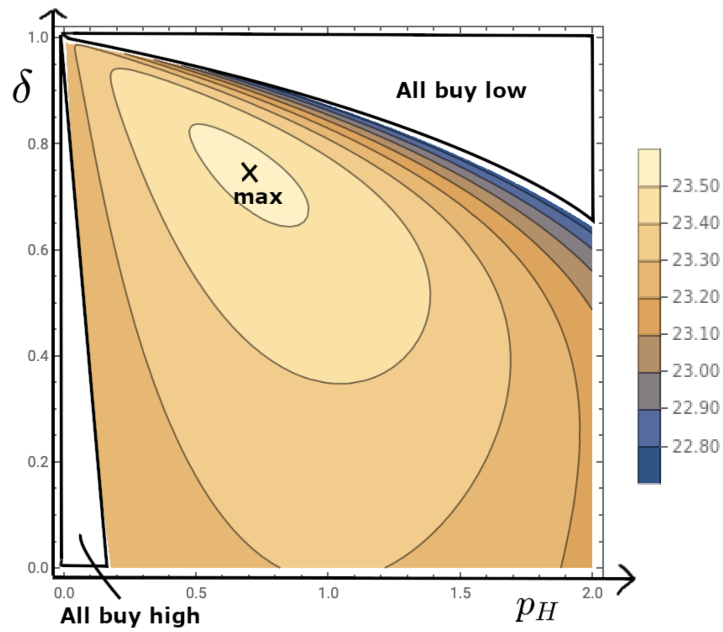
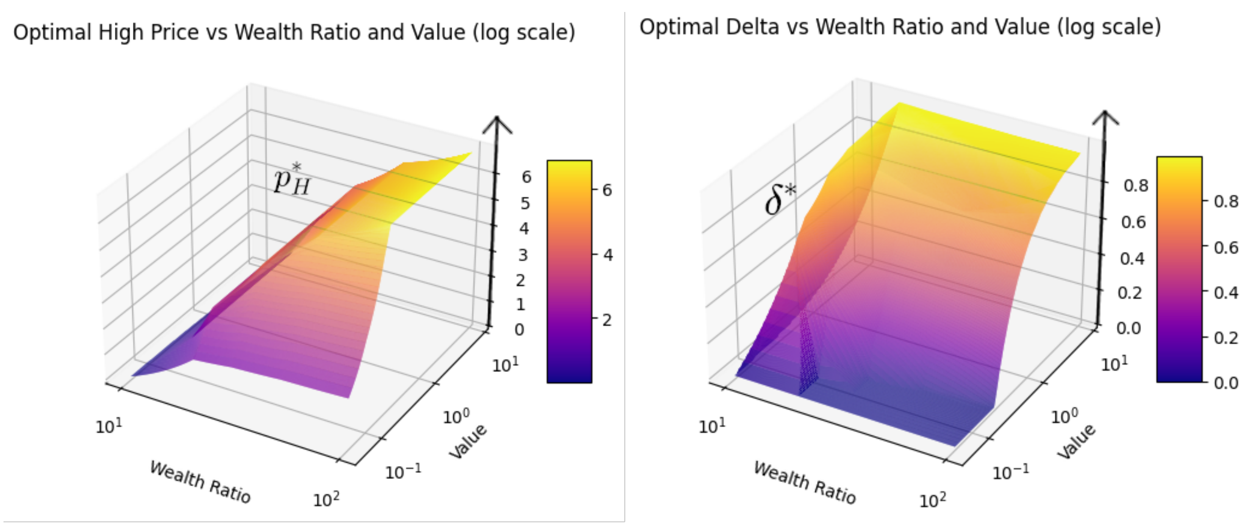


Figure 4: This figure shows a contour plot of the objective function for different values of the choice parameters p_H and δ. The exogenous parameters for this plot are $\underline{w} = 0.1$, $\overline{w} = 10$, and $v = 1$. δ ranges from 0 to 1 and p_H ranges from 0 to 2. The optimal choice parameters here are $p_H = 0.689$ and $\delta = 0.75$

take the low price (so optimal $\delta = 1$) or purchase at the high price (so $r = p_H$ and everyone gets their payment back). In either case, all consumers receive the good for free and there is no rationing or redistribution. Looking at figure (4), an interior solution exists for these exogenous parameters at $p_H^* = 0.689$ and $\delta^* = 0.75$, the point on the figure that is marked with an x. When there are infinitely many consumers with uniformly distributed wealth, it is not immediately obvious that an interior solution like this should exist. However, I demonstrate the same counterintuitive result in the uniform case as the one from the Bob and Elon problem where the parameters that optimize the utilitarian social welfare function result in an allocation that is not ex-post Pareto efficient.

3.2.1 The Optimal Parameters

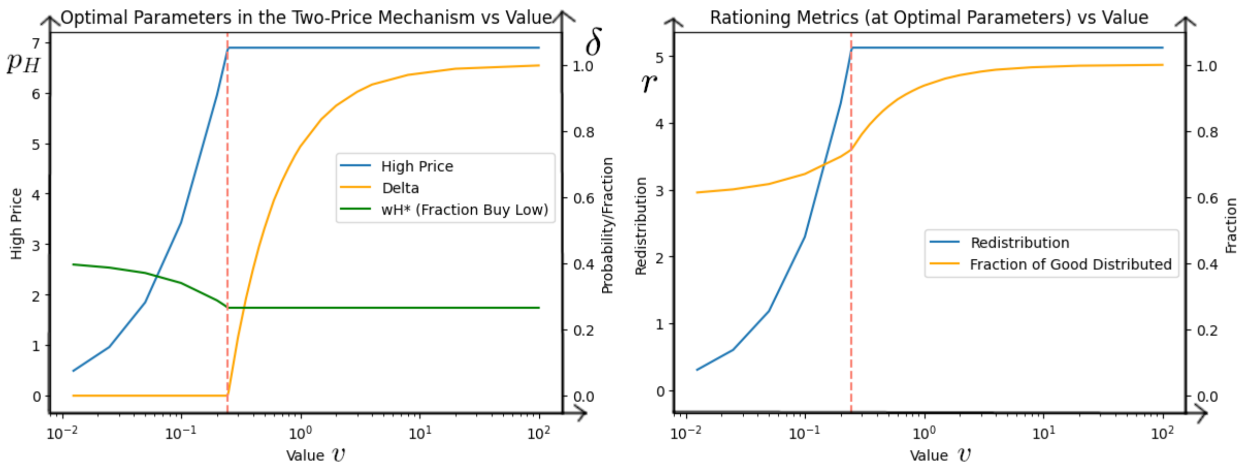
I can continue performing this analysis for various values of the exogenous parameters to get a sense of what the solutions look like. Like the Bob and Elon example, when v is fixed, p_H^* and δ^* only depend fundamentally on the ratio $\frac{\overline{w}}{\underline{w}}$, which will be defined as the wealth ratio for the rest of this section. δ^* is constant with respect to the wealth ratio while p_H^* is scaled by the absolute wealth level. For example, if $\underline{w}$ and $\overline{w}$ both doubled, δ^* would be unaffected but p_H^* would double. The dependence of the optimal parameters on the wealth ratio seems like a relic of the functional form of the utility function being $\ln(w)$ and its property of constant relative risk aversion. Because the optimal parameters only depend on two exogenous variables, I plot the optimal parameters in 3D space as a function of the wealth ratio and the value. The results of this analysis are shown in figure 5.



(a) Optimal High Price (b) Optimal Delta

Figure 5: This figure shows the optimal values for p_H and δ for different values of the two exogenous variables: the wealth ratio $\frac{\overline{w}}{\underline{w}}$ and the value v. The x and y axes are on a log scale.

I hold $\overline{w}$ fixed at 100 for the remainder of this section so $\underline{w}$ is adjusted to get a certain wealth ratio. As seen in figure 5a, p_H^* is increasing in the wealth ratio but plateaus relatively quickly as the wealth ratio becomes large. If the poorest consumer becomes poorer, then the marginal gains to redistribution increase and so the high price should increase to enable a greater amount of redistribution. δ^* is decreasing in the wealth ratio but is much more affected by the value v than the wealth ratio. Because redistribution is relatively more valuable when the wealth ratio is high, δ^* should decrease as it enables a higher level of redistribution. If δ is high, then many consumers are disincentivized from purchasing the good at the high price, preferring to take the δ probability to win the good for free. Because the behavior of the optimal parameters looks relatively similar while fixing the wealth ratio, I plot at a cross-section of figure 5 to more easily visualize the relationship between the optimal parameters and a consumer's value. This is shown in figure 6.



(a) Optimal Parameters and w_H^*

(b) Rationing Metrics

Figure 6: Fixing $\overline{w}$ at 100 and $\underline{w}$ at 1 (so the wealth ratio is 100), (a) shows p_H^*, δ^*, and w_H^* as a function of the value on a log scale. w_H^* is represented here in terms of the fraction of consumers who trade at the low price $\left(\text{so } \frac{w_H^* - \underline{w}}{\overline{w} - \underline{w}}\right)$. (b) shows r and the fraction of consumers who receive the good (in expectation) as a function of the value on a log scale. The fraction of consumers who receive the good in expectation is defined as the fraction of consumers who purchase at the high price * 1 + the fraction of consumers at the low price * δ. The pink dotted line marks the kinked behavior of the optimal parameters and this occurs between $v = 0.245$ and $v = 0.25$.

Fixing the wealth ratio at 100, figure 6 shows the optimal parameters and some rationing metrics as a function of the value. Notice the similarities of this plot versus the analogous plot for the Bob and Elon problem in figure 2. The optimal parameters for the uniform case qualitatively look the same as they do in the Bob and Elon problem, leading to a similar proposition describing their behavior. Define $\left(\frac{\overline{w}}{\underline{w}}\right)^*$ as the minimum wealth ratio at which rationing becomes optimal in the uniform case.

Proposition 3. *In the case with an infinite number of consumers with uniformly distributed wealth, for a fixed wealth ratio greater than* $\left(\frac{\overline{w}}{\underline{w}}\right)^* \approx 13$*, there exists a cutoff value* v^* *where* *the behavior of the optimal parameters shifts. For $v \leq v^*$, p_H^* is increasing in v and δ^* is set to 0. For $v > v^*$, p_H^* is fixed and δ^* asymptotically approaches 1 as $v \to \infty$.*

As seen in figure 6a, the optimal parameters exhibit kinked behavior at some cutoff value v^* (between $v = 0.245$ and $v = 0.25$). Before v^*, the high price increases as the value increases while δ is held at 0 and after v^*, the high price is held constant and δ asymptotically approaches 1. Intuitively, when the value is low, the marginal benefit of increasing δ is low relative to the marginal benefit of increasing redistribution since consumers do not value the good very much. However, when the value is low, the high price must also be low to induce redistribution because consumers have a low willingness to pay for the good. Before the kink, w_H^* decreases as v increases so more consumers are buying at the high price even though w_H^* is increasing in p_H. This is because w_H^* is decreasing in v so if consumers value the good more, they are more willing to pay for the good at the high price and this more than offsets the effect of a higher price. After the kink, w_H^* is fixed so the same fraction of consumers trades at the high price versus the low price.

Because p_H^*, w_H^*, and r are all fixed after the cutoff value, I can get an analytical solution for δ as a function of v.

Proposition 4. *For $v > v*$, p_H^*, w_H^*, and r are fixed and $\delta^*(v) = 1 - \frac{c}{v}$ where c is a positive constant and is a function of p_H^* and w_H^*: $c = \ln(w_H^* + r(p_H, w_H^*)) - \ln(w_H^* - p_H^* + r(p_H, w_H^*))$.*[2]

Proof. First, the indifference condition for the consumer at w_H^* from (14) must hold. To reiterate, the consumer with wealth w_H^* is indifferent between the low price and the high price, so: $\delta v + \ln(w_H^* + r) = v + \ln(w_H^* - p_H + r)$. Solving for δ, I get:

$$\delta^*(v) = 1 - \frac{\ln(w_H^* + r) - \ln(w_H^* - p_H + r)}{v} \text{ for } v \geq v^* \tag{25}$$

Because w_H^*, r, and p_H^* are all constant after the cutoff value, the numerator of the fraction is also constant. □

2. r is fully determined by p_H^* and w_H^* so it is not an independent variable.

This analytical form is reminiscent of δ^* in (4) from the Bob and Elon problem. Recall the solution from that problem: $\delta^* = 1 + \frac{1}{v} \ln\left(\frac{2(w_H + w_L)}{3w_H}\right)$ if $\frac{w_H}{w_L}(3e^{-v} - 2) < 2$ and $\frac{w_H}{w_L} \leq 2$. For the Bob and Elon problem, if rationing is optimal, then $\delta^* < 1$. And if w_H and w_L are held fixed, then δ^* would also look like $1 - \frac{c}{v}$. In the Bob and Elon problem, $w_H^* = w_H$ to make Elon indifferent between the low price and the high price in order to maximize redistribution if rationing were optimal. As w_H^* is held constant in the continuous case, I obtain a very similar expression for the optimal δ as a function of v.

The solution is kinked at the cutoff value because $\delta \geq 0$. If there were no restrictions on δ, then the mechanism would set negative values for δ before the kink with the same constant p_H and w_H^* for all v. This is qualitatively identical to the solutions from the Bob and Elon problem where if the wealth ratio is fixed and δ^* is unrestricted, then p_H^* and w_H^* are constant for all v and $\delta^* < 0$ when $v < v^*$.

Comparing the value of w_H^*, the fraction of consumers in the Bob and Elon problem that gets a raffle ticket is 50% while the fraction of consumers in the uniform case that gets a raffle ticket for $v > v^*$ is 26%. This reflects the fact that rationing is relatively less valuable in the uniform case because the wealth distribution is less unequal. When there are an infinite number of consumers that are uniformly filled in between Bob and Elon, the consumers that are poorer than Elon that trade at the high price are hurt more since their marginal utility for money is larger than Elon's while the consumers who are wealthier than Bob that get a raffle ticket receive a smaller utility benefit from redistribution. The mechanism in the Bob and Elon problem is able to concentrate the entire revenue burden onto Elon by making his incentive-compatibility constraint bind while concentrating half of the redistribution on Bob because he is one of two agents. This concentrated cost and benefit allows the mechanism to more precisely distribute wealth from Elon to Bob. Because rationing is relatively less valuable in the uniform case, this also makes the conditions for rationing stricter than the conditions for the Bob and Elon problem. For the uniform case, the wealth ratio $\left(\frac{\overline{w}}{\underline{w}}\right)$ must be greater than 13 for rationing to be optimal versus the Bob and Elon problem where the wealth ratio $\left(\frac{w_H}{w_L}\right)$ must be greater than 2.

Looking at the rationing metrics, it makes sense that the fraction of consumers who receive the good in expectation is strictly increasing in the value if the wealth ratio is fixed since more consumers should receive the good as they value the good more, as else equal.

3.2.2 The Cutoff Value

I can unfix the wealth ratio and observe the cutoff value v^* as a function of the wealth ratio. Recall from the Bob and Elon problem (Proposition 2), the cutoff value $v^* = \ln\left(\frac{3\frac{w_H}{w_L}}{2\left(\frac{w_H}{w_L}+1\right)}\right)$ so I can compare the cutoff values for these two cases. Define the cutoff value for the Bob and Elon problem as v_B^* and the cutoff value for the uniform case as v_U^*. Figure 7 plots v_B^* and v_U^* versus the respective wealth ratios.

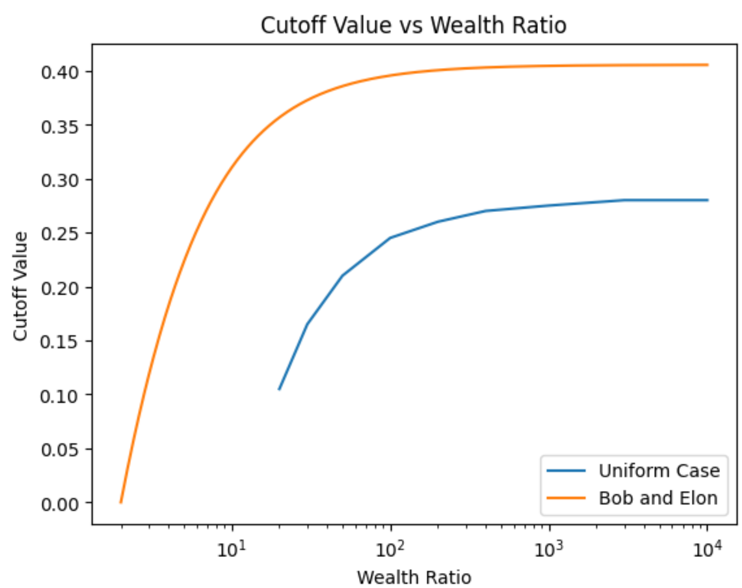


Figure 7: This figure shows the cutoff value versus the wealth ratio in the uniform case (v_U^*) and the Bob and Elon problem (v_B^*). The wealth ratio for the uniform case is defined as $\frac{\overline{w}}{\underline{w}}$ while the wealth ratio for the Bob and Elon problem is defined as $\frac{w_H}{w_L}$. The data points for the uniform case range from $\frac{\overline{w}}{\underline{w}} = 20$ to $\frac{\overline{w}}{\underline{w}} = 10000$ and the plotted function for the Bob and Elon problem range from $\frac{w_H}{w_L} = 2$ to $\frac{w_H}{w_L} = 10000$.

Proposition 5. *The cutoff value $v^* = 0$ at $\frac{w_H}{w_L} = \left(\frac{w_H}{w_L}\right)^*$ or $\frac{\overline{w}}{\underline{w}} = \left(\frac{\overline{w}}{\underline{w}}\right)^*$ and is strictly increasing in the wealth ratio.*

The cutoff value for the uniform case only begins at $\frac{\overline{w}}{\underline{w}} = 20$ because rationing is optimal for a smaller range of wealth ratios and this is the smallest data point I had.[3] Presumably, v_U^* should continue to decrease as the wealth ratio decreases and reach 0 at $\frac{\overline{w}}{\underline{w}} = \left(\frac{\overline{w}}{\underline{w}}\right)^*$.

The cutoff value can be interpreted as the point at which the benefit of increasing δ exceeds the benefit of increased redistribution. Note that the cutoff value is increasing in the wealth ratio. This makes sense because redistribution is relatively more valued as the wealth ratio increases.

Notice that v_B^* is strictly greater than v_U^* for all wealth ratios. This is because redistribution is relatively more valuable in the Bob and Elon case versus the uniform case because the mechanism can be more precise in redistributing wealth from Elon to Bob so v^* should be higher in the Bob and Elon case.

It may seem that the wealth ratio in the Bob and Elon problem cannot be fairly compared to the wealth ratio in the uniform case and that the result that $v_B^* > v_U^*$ is obvious based on the setup. However, even if there were some scaling factor to fairly compare these two wealth ratios, it is clear that the asymptotic behavior of the cutoff value differs: $v_B^* > v_U^*$ as $v \to \infty$.

3.2.3 Discussion

In the case of infinitely many consumers with uniformly distributed wealth, the main themes of the Bob and Elon problem reappear. I rederive the counterintuitive result that even though the good is plentiful, it may be rationed in order to induce incentive-compatible redistribution. Also, for a sufficiently large wealth ratio, rationing is always optimal and all consumers only receive the good in the limit where $v \to \infty$.

The solutions for this problem are qualitatively very similar to the solutions from the Bob and Elon problem. When the wealth ratio exceeds some critical point $\left(\frac{\overline{w}}{\underline{w}}\right)^* \approx 13$, rationing

3. Numerically maximizing at wealth ratios close to $\left(\frac{\overline{w}}{\underline{w}}\right)^*$ was challenging as the software could not detect very small changes in the objective function.

becomes optimal. Then for $v \leq v^*$, p_H^* is increasing in v and δ^* is set to 0 and for $v > v^*$, p_H^* is fixed and δ^* asymptotically approaches 1 as $v \to \infty$ (Proposition 3). The main difference I find is that the conditions for rationing are stricter in the uniform case than in the two consumer case. For the uniform case, $\left(\frac{\overline{w}}{\underline{w}}\right)^* \approx 13$ while for the Bob and Elon problem, $\left(\frac{w_H}{w_L}\right)^* = 2$. This is because the mechanism can more precisely redistribute wealth in the Bob and Elon problem and this results in redistribution being relatively more valuable. This also leads to a lower w_H^* in the uniform case versus the Bob and Elon problem so in the uniform case, the good is cheaper and more consumers purchase it at the high price.

These rationing conditions do not align with the result from Dworczak, Kominers, and Akbarpour (2021) that finds that rationing is optimal when the marginal utility for money of the poorest consumer is larger than twice the marginal utility for money of the median consumer in the case where agents' marginal rates of substitution are uniformly distributed. Although I use a slightly different mechanism from the one in their paper, because $p_L = 0$, these two mechanisms act the same since a δ probability of paying the low price is the same as paying 0. In this setting, rationing becomes optimal when $\frac{\overline{w}}{\underline{w}} > \left(\frac{\overline{w}}{\underline{w}}\right)^*$ so at $\left(\frac{\overline{w}}{\underline{w}}\right)^* \approx 13$, the median consumer has 7 times as much wealth as the poorest consumer. With a log utility function, this means the poorest consumer's marginal utility for money is 7 times as large as the median consumer. This disparity in the rationing conditions arises from differing assumptions. Dworczak, Kominers, and Akbarpour assume that consumers' marginal rates of substitution are uniformly distributed while I assume that wealth is uniformly distributed. The rate of substitution is defined as the marginal value of the good over the marginal value of money. In this setting, all consumers have the same marginal value for the good v but their marginal value of money is $\frac{1}{w}$. If w is uniformly distributed, then the distribution of $\frac{1}{w}$ has a rapid drop-off from its peak at $x = \frac{1}{\overline{w}}$, with a long tail on the right-hand side. This means that there are many more consumers with a high rate of substitution or a correspondingly low marginal utility for money compared to the case that Dworczak, Kominers, and Akbarpour examined. So rationing is not as effective since there is a higher density of consumers with low marginal utilities for money and increasing redistribution only helps a small number of consumers. Additionally, Dworczak, Kominers, and Akbarpour arrive at their result of the rationing conditions through an analysis of rationing at the margin. However, because of the assumption of uniformly distributed wealth, rationing at the margin always reduces total utility, only becoming net positive with further rationing. So the conditions for rationing are stricter in this setting since additional rationing must offset the initial losses to total utility.

4 Policy Implications

Real world redistributive policies are often seen as a way to achieve more equitable outcomes at the cost of distorting the Marshallian efficient competitive equilibrium. However, the results of this paper show that the competitive equilibrium does not necessarily maximize a utilitarian social welfare function if the assumption of quasilinearity is removed. When consumers experience wealth effects and there is a significant disparity in wealth, redistribution and rationing through the two-price mechanism become optimal. This model assumed that wealth was uniformly distributed; however, the wealth/income distribution in the real world is far from uniform and has a long right tail (Piketty 2014). Although, it does not seem that this would change the main results of the model so it seems reasonable to apply this mechanism to this case.

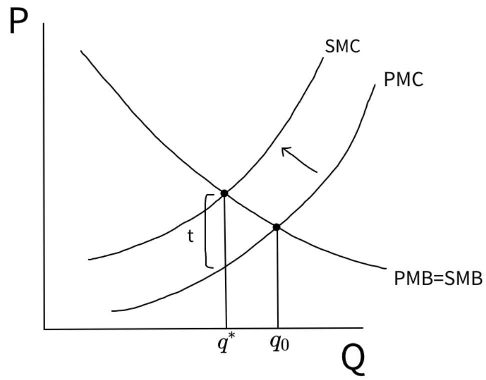
There are no obvious implementations of this mechanism in reality. This mechanism may not be politically feasible over ethical concerns for a policy that rations the poorest consumers. One could view the housing market in the US as a crude approximation of the two-price mechanism. The US Department of Housing and Urban Development (HUD) has a public housing program that provides affordable rental housing to eligible low-income families, the elderly, and people with disabilities. People interested in this program must apply and if accepted, are put on a waiting list. This program can be viewed as poorer consumers paying a lower price compared to the open market and getting some probability of obtaining the rental unit, in this case, they have to wait on the waiting list for an indeterminate amount of time. On the other hand, wealthier consumers can choose to trade on the open market and avoid the waiting list by paying a higher price. Moreover, the price on the open market is inflated relative to the true costs of production due to government-levied property taxes. These taxes are then used for general government spending but hopefully some of it goes into building infrastructure or subsidizing affordable housing and is redistributed back to the citizens. This is an imperfect analogy for the two-price mechanism since the government is able to condition public housing on income. With the ability to price discriminate, there is no longer an incentive-compatibility constraint because wealthier consumers cannot gain access to the public housing program.

What other real-world markets could this model be applied to? In general, socially beneficial redistribution would be under the purview of the government and not private corporations. Additionally, this mechanism breaks down in the presence of competition since the high price would presumably be greater than the marginal cost of production so it would be outcompeted by another firm that produces at the marginal cost. Charging the high price to obtain positive profits for redistribution is only possible if there is no better outside option for consumers. So it would make the most sense for the good to be managed by the government instead of the private sector. Additionally, to combat competition issues, the government should have a monopoly over the supply or access to the good. One area that fits these assumptions is congestion pricing. First, I illustrate how the two-price mechanism might work in a generalized setting for a good with a negative externality.

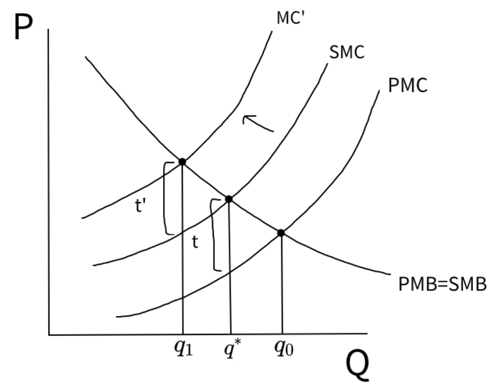
4.1 A Market with a Negative Externality

This section shows how the two-price mechanism dominates a simple Pigouvian tax in a marketplace with a negative externality on consumption, illustrated by figure 8.

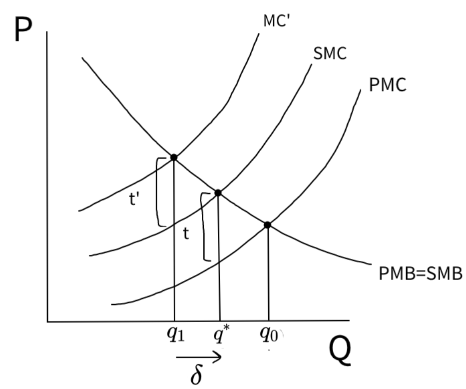
The effect of the two-price mechanism in this market can be imagined in three steps. Figure 8a shows the usual story of a marketplace with a negative externality. Social marginal



(a)



(b)



(c)

Figure 8: This figure shows a marketplace for a good with a negative externality on consumption and the three steps of the two-price mechanism that brings this market to q^*. There is a Pigouvian tax t that corrects the negative externality since social marginal cost exceeds private marginal cost, an additional redistributive tax t', and then the low price raffle (δ) is used to reach the socially optimal quantity q^*.

cost exceeds private marginal cost and there is more consumption than the socially optimal quantity ($q_1 > q^*$). The usual solution to this problem is to impose a Pigouvian tax t such that the private marginal cost curve plus the tax intersects the marginal benefit curve at q^*, reaching the socially desirable quantity. However, this paper has shown that if consumers have different marginal utilities for money, it may be optimal to ration the good further and increase the price past the competitive equilibrium price to acquire more revenue to redistribute. In figure 8b, an additional redistributive tax t' is implemented[4] that increases

4. I assume that the private marginal cost and social marginal cost are separated by some constant so

the price further in order to acquire more profit. However, the quantity becomes q_1, which is smaller than the social optimum. To achieve the socially optimal quantity q^*, I can then use the low price to hold a lottery for the good where consumers have probability δ to win the good[5], shown in figure 8c. One thing that is omitted from figure 8 is that consumers' behavior would change after introducing the low price since this would make paying for the original good less attractive. This would change the demand curve since there is now a more attractive outside option which would decrease the quantity demanded. However, if I set the parameters of the mechanism correctly, I should be able to account for this effect and reach q^*.[6] The next section discusses congestion pricing and how the two-price mechanism might apply in this setting.

4.2 Congestion Pricing

Congestion pricing has long been supported in the theoretical economic literature for its benefits to allocative efficiency. Drivers only consider their private marginal cost of driving which results in more drivers on the road than is socially optimal. In order to remedy this, many economists propose a Pigouvian tax on congestion so that drivers internalize the negative externality they pose to others on the road. Beckmann, McGuire, and Winsten (1955) argue that optimal transportation pricing policies should reflect the full social costs of transportation, including infrastructural, environmental, and congestion costs. The classical graph-theoretic analysis of congestion pricing essentially finds that implementing a Pigouvian tax on each edge minimizes the total latency (sum of travel times), assuming that all agents are homogeneous in how they trade off money and time. Cole, Dodis, and Roughgarden (2003) extend this model in a way that is relevant to this paper. They examine agents with

that I can impose a total tax of $t + t'$.

5. q^* is only optimal if those with the highest willingness to pay obtain the good and a lottery would not achieve that. However, if I assume that all consumers have the same value for the good and their willingness to pay only reflects their marginal utility of money, then this justifies this quantity as socially optimal even with a lottery.

6. Presumably, this means that δ would sometimes be set to 0 so the two-price mechanism would turn into a standard Pigouvian tax.

heterogeneous preferences for time and money and find that the edges of a network can always be priced such that an optimal routing of traffic arises as a Nash equilibrium.

Singapore famously put congestion pricing into practice in 1975 with the Area Licensing Scheme (ALS) to deal with traffic congestion problems. They defined a restricted zone (RZ) in the central business district where all non-exempt vehicles would have to purchase a permit in order to access the RZ between 7:30 a.m. to 9:30 a.m. Monday through Saturday (Phang and Toh 2006).

In theory, a Pigouvian tax should restore allocative efficiency; however, the results of this paper show that in the presence of wealth effects, this equilibrium may still not be optimal from a utilitarian perspective as there are additional gains to redistribution that can be obtained by charging wealthier consumers a higher price for access to the road. In this setting, I don't view congestion pricing solely in the traditional Pigouvian sense but I use the consumer's time as a commonly valued good that enables incentive-compatible redistribution. Imagine the redistribution is accomplished not through an equal lump-sum payment to all consumers but perhaps investment in infrastructure or public transportation.

To implement the two-price mechanism in reality, I draw inspiration from the ALS in Singapore. Imagine charging a high price to access a road during peak hours as well as holding a free lottery to raffle off access to the road with some probability. Access to a road is a suitable good to apply this mechanism to because it is indivisible and consumers would have unit demand for this good which are the main assumptions of this model. Additionally, a consumer's willingness to pay for the road is a proxy for their value for time which is more indicative of their marginal utility for money than any inherent individual preferences.

One issue with this application is that this model does not take into account the negative externality that is posed by additional drivers on the road. The supply of "access to the road" is more constrained by the negative externality of congestion than by an actual supply constraint. I modify the Bob and Elon model to incorporate this negative externality and find similar results to the original model, justifying this application. The derivation and results are shown in the appendix.

Congestion pricing is traditionally seen solely as a solution to a negative externality. However, based on the results of the modified model, access to the road can be sold to induce welfare-enhancing redistribution in addition to solving the negative externality, and I want to hold this lottery that gives Bob some probability to drive on the road. Holding a lottery in the context of congestion pricing seems counterintuitive but this model shows that this is optimal, at least in this setting.

5 Conclusion

This paper investigates consumers with non-quasilinear utility and applies the two-price mechanism as a tool to redistribute wealth in an incentive-compatible way. Using a utilitarian social welfare function, I find that the two-price mechanism dominates the "competitive equilibrium" of giving the good away for free when the consumers have sufficiently different marginal utilities for money. There is a loss to allocative efficiency when the price is set higher than 0 but differences in consumers' marginal utilities for money can make redistribution sufficiently desirable to where a price greater than 0 is optimal. Due to incentive-compatibility constraints, this necessarily entails a probability that some consumers don't receive the good even when it is plentiful.

I find that rationing becomes optimal in a two-consumer (Bob and Elon) model when Bob's marginal utility for money is at least twice as large as Elon's marginal utility for money for any utility function for wealth that is monotonic and strictly concave. For a log utility function, this means that Elon's wealth must be at least twice as large as Bob's wealth. With infinitely many consumers with uniformly distributed wealth, the wealth ratio at which rationing becomes optimal is around 13.

I also describe the behavior of the solutions to the Bob and Elon problem and the uniformly distributed case. If rationing is optimal, then for $v < v^*$, p_H^* is increasing in v and δ^*

is set to 0. And for $v > v^*$, p_H^* is fixed and δ^* asymptotically approaches 1. I also examine the behavior of the cutoff value $v*$ in the Bob and Elon problem and the uniform case.

With wealth effects, there become many scenarios in which increasing the price above marginal cost and redistributing excess profits enhances total welfare as long as there exists a commonly valued good that can be monopolized. To illustrate, I discuss how the two-price mechanism is applicable to congestion pricing and modify the original Bob and Elon model to account for the negative externality in congestion.

The effects of increased redistribution on long-term incentives are not discussed in this paper. It may be the case that a system with increased redistribution discourages the effort to earn a higher income and results in slower long-term economic growth as productivity decreases. This concern is outside of the scope of this paper but as stated in the introduction, Stiglitz (2016) argues that decreasing inequality in the US can actually enhance long-term economic performance. This mechanism can result in higher total utility than the competitive equilibrium while also creating a more equitable allocation of wealth which is doubly desirable.

In the absence of externalities or distortionary market power, many economists treat the competitive equilibrium as the optimal allocation of goods. However, this paper shows that in the presence of wealth effects, which are clearly exhibited in the real world, the competitive equilibrium may not maximize a utilitarian social welfare function. When consumers have sufficiently different marginal utilities for money, the benefit from redistribution exceeds the cost to allocative efficiency and the two-price mechanism results in greater total utility than giving the plentiful good away for free.

References

Akbarpour, Mohammad, Piotr Dworczak, and Scott Duke Kominers. "Redistributive Allocation Mechanisms." *SSRN Electronic Journal*, 2020. ISSN: 1556-5068. https://www.ssrn.com/abstract=3609182.

Baisa, Brian. "Auction design without quasilinear preferences." *Theoretical Economics* 12, no. 1 (2017): 53–78. ISSN: 1555-7561. https://onlinelibrary.wiley.com/doi/abs/10.3982/TE1951.

Beckmann, Martin, C.B. McGuire, and Christopher Winsten. *Studies in the Economics of Transportation*. Rand Corporation, 1955.

Cole, Richard, Yevgeniy Dodis, and Tim Roughgarden. "Pricing Network Edges for Heterogeneous Selfish Users." In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, 521–530. STOC '03. New York, NY, USA: Association for Computing Machinery, June 9, 2003. ISBN: 978-1-58113-674-6. https://dl.acm.org/doi/10.1145/780542.780618.

Dworczak, Piotr, Scott Duke Kominers, and Mohammad Akbarpour. "Redistribution Through Markets." *Econometrica* 89, no. 4 (2021): 1665–1698. ISSN: 0012-9682. https://www.econometricsociety.org/doi/10.3982/ECTA16671.

Horowitz, John, J. List, and K. E. McCONNELL. "A Test of Diminishing Marginal Value." *Economica* 74, no. 296 (2007): 650–663. ISSN: 1468-0335. https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1468-0335.2006.00565.x.

"HUD's Public Housing Program." HUD.gov / U.S. Department of Housing and Urban Development (HUD), September 20, 2017. https://www.hud.gov/topics/rental_assistance/phprog.

Kazumura, Tomoya, Debasis Mishra, and Shigehiro Serizawa. "Mechanism design without quasilinearity." *Theoretical Economics* 15, no. 2 (2020): 511–544. ISSN: 1555-7561. https://onlinelibrary.wiley.com/doi/abs/10.3982/TE2910.

Myerson, Roger B, and Mark A Satterthwaite. "Efficient mechanisms for bilateral trading." *Journal of Economic Theory* 29, no. 2 (April 1983): 265–281. ISSN: 00220531. https://linkinghub.elsevier.com/retrieve/pii/0022053183900480.

Phang, Sock-Yong, and Rex S. Toh. "Road Congestion Pricing in Singapore: 1975 to 2003." Publisher: Penn State University Press, *Transportation Journal* 43, no. 2 (2004): 16–25. ISSN: 0041-1612. https://www.jstor.org/stable/20713563.

Piketty, Thomas. *Capital in the Twenty-First Century.* Harvard University Press, 2014.

Pycia, Marek, and Rodney Garratt. "Efficient Bilateral Trade." In *Proceedings of the The Third Conference on Auctions, Market Mechanisms and Their Applications.* The Third Conference on Auctions, Market Mechanisms and Their Applications. Chicago, United States: ACM, 2015. ISBN: 978-1-63190-075-4. http://eudl.eu/doi/10.4108/eai.8-8-2015.2260906.

Stiglitz, Joseph E. "Inequality and Economic Growth," 2016, 134–155. https://doi.org/10.7916/d8-gjpw-1v31.

Weitzman, Martin L. "Is the Price System or Rationing More Effective in Getting a Commodity to Those Who Need it Most?" Publisher: [RAND Corporation, Wiley], *The Bell Journal of Economics* 8, no. 2 (1977): 517–524. ISSN: 0361-915X. https://www.jstor.org/stable/3003300.

Appendix

Second Order Conditions for the Original Bob and Elon Problem

To test if the solution is a maximum of the objective function, I check the SOCs. From the incentive-compatibility constraint in (1), I find that $p_H = \frac{2w_H(e^{v(1-\delta)}-1)}{e^{v(1-\delta)}}$. I can then plug this into the maximization problem in (2), turning the multivariate optimization problem into a single variable one. So the new maximization problem is

$$\max_{\delta} \left[\delta v + \ln\left(w_L + \frac{w_H\left(e^{v(1-\delta)}-1\right)}{e^{v(1-\delta)}}\right) + v + \ln\left(w_H - \frac{w_H\left(e^{v(1-\delta)}-1\right)}{e^{v(1-\delta)}}\right)\right]$$

The first order condition is

$$\frac{v\left(3e^{\delta v}w_H - 2e^{v}(w_H + w_L)\right)}{e^{\delta v}w_H - e^{v}(w_H + w_L)} = 0$$

which is satisfied for the solution of $\delta^* = 1 + \frac{1}{v}\ln\left(\frac{2(w_H+w_L)}{3w_H}\right)$ from (4).

And the second order condition is

$$-\frac{e^{v(1+\delta)}v^2 w_H(w_H + w_L)}{(e^{\delta v}w_H - e^{v}(w_H + w_L))^2}$$

This second order condition is globally negative which makes this problem globally concave and confirms the solution as a global maximum.

The Bob and Elon Model with a Negative Externality

Continuing with the example of congestion pricing, the good that the mechanism designer would be selling could be access to a certain road during rush hour. Suppose that Bob's consumption of the good negatively affects Elon's enjoyment of the good and vice versa, incorporating the fact that drivers cause a negative externality on other drivers through increased congestion. I model each agent's value for the good as a function of the other agent's consumption: $v(q_{-i}) = v(1 - cq_{-i})$ where v represents the value of the road without any other drivers and $0 \leq c \leq 1$ is a constant that represents the magnitude of disutility of additional drivers on the road. For example, if $c = \frac{1}{2}$, then the other person's usage of the road makes the road half as valuable for you. And if $c = 0$, the initial Bob and Elon problem is recovered where there is not a negative externality on consumption.

Each agent's utility is given by: $u_i(q_i, q_{-i}, w_i) = q_i v(1-cq_{-i}) + \ln(w_i)$ where $q_i, q_{-i} \in [0, 1]$. Using the same reasoning from the original Bob and Elon problem, optimal $p_L = 0$ because I never want to redistribute wealth from Bob to Elon. I also assume that there exists a rationing solution so Bob trades at the low price and Elon trades at the high price.

Their total utility in expectation is $V = \delta v_B(q_E) + \ln(w_L + r) + v_E(q_B) + \ln(w_H - p_H + r)$ where $v_B(q_E)$ is Bob's value for the road given Elon's consumption and $v_E(q_B)$ is Elon's value for the road given Bob's consumption. Recall $r = \frac{p_H}{2}$ since the total revenue from the mechanism is p_H and $q_B = \delta$ and $q_E = 1$. So the objective function simplifies to

$$V = \delta v(1 - c) + \ln\left(w_L + \frac{p_H}{2}\right) + v(1 - \delta c) + \ln\left(w_H - \frac{p_H}{2}\right)$$

q_B is a random variable but I can plug in $q_B = \delta$ because Elon's value function is linear in q_B so that would be his value function in expectation.

If rationing is optimal, I want to maximize redistribution given some δ so Elon should be indifferent between the low price and the high price: $\delta v(1 - cq_B) + \ln(w_H) = v(1 - cq_B) + \ln\left(w_H - \frac{p_H}{2}\right)$ where $q_B = \delta$. Solving, Elon is indifferent between the low price and high price if

$$w_H = \frac{e^{v(1-\delta)(1-\delta c)} p_H}{2\left(e^{v(1-\delta)(1-\delta c)} - 1\right)}$$

I want to then maximize total utility subject to Elon's incentive-compatibility constraint:

$$\max_{p_H, \delta} \delta v(1-c) + \ln\left(w_L + \frac{p_H}{2}\right) + v(1-\delta c) + \ln\left(w_H - \frac{p_H}{2}\right) \;\; s.t. \; w_H = \frac{e^{v(1-\delta)(1-\delta c)} p_H}{2\left(e^{v(1-\delta)(1-\delta c)} - 1\right)} \quad (26)$$

There does not seem to exist an analytical solution for a general c but I can obtain explicit expressions for the unrestricted optimal parameters for $c = \frac{1}{2}$.

$$(p_H^*)^{MOD} = w_H - w_L$$

$$(\delta^*)^{MOD} = \frac{3}{2} - \frac{1}{2}\sqrt{1 + \frac{8}{v}\ln\left(\frac{2w_H}{w_H + w_L}\right)}$$

Restricting $(\delta^*)^{MOD}$ to be between 0 and 1, I get the piecewise form of the optimal parameters. $(\delta^*)^{MOD} \geq 1$ only if $w_H \leq w_L$ which is false by assumption so $(\delta^*)^{MOD}$ will always be less than 1.

$$(\delta^*)^{MOD}(w_H, w_L, v) = \begin{cases} 0 & \text{if } \frac{w_H}{w_L}(2e^{-v} - 1) > 1 \\ \frac{3}{2} - \frac{1}{2}\sqrt{1 + \frac{8}{v}\ln\left(\frac{2w_H}{w_H+w_L}\right)} & \text{if } \frac{w_H}{w_L}(2e^{-v} - 1) < 1 \end{cases} \tag{27}$$

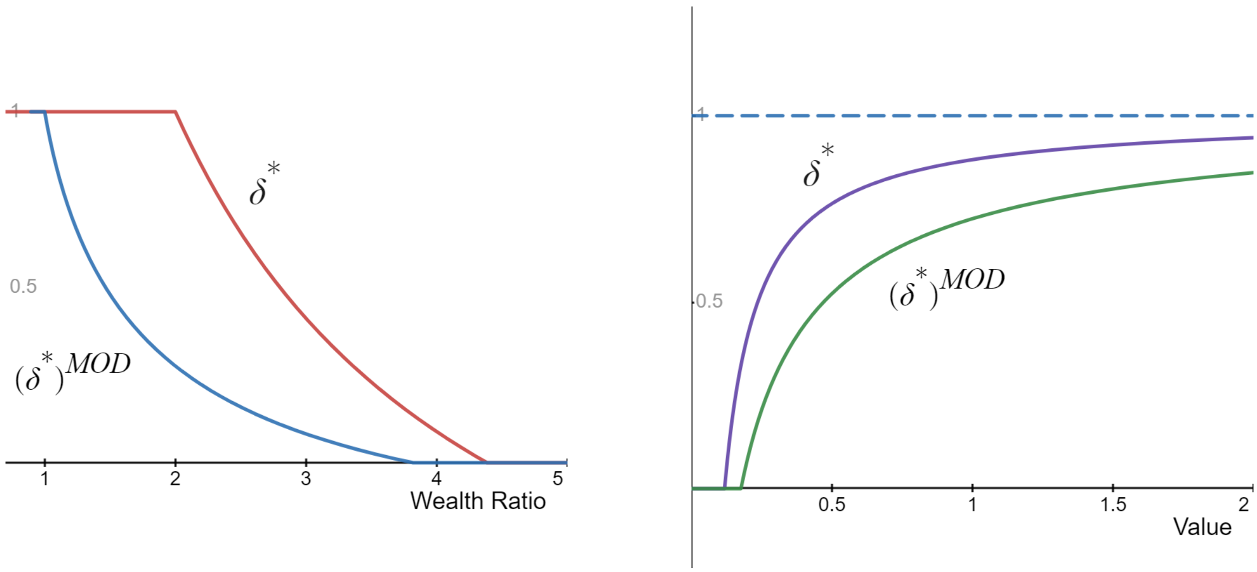
$$(p_H^*)^{MOD}(w_H, w_L, v) = \begin{cases} 2w_H\left(1 - \frac{1}{e^v}\right) & \text{if } \frac{w_H}{w_L}(2e^{-v} - 1) > 1 \\ w_H - w_L & \text{if } \frac{w_H}{w_L}(2e^{-v} - 1) < 1 \end{cases} \tag{28}$$

Second Order Conditions

I can verify the second order conditions of the problem using the same strategy as in the original Bob and Elon problem by converting the two variable optimization problem into a single variate one. In the indifference condition, I get that $p_H = \frac{2w_H\left(e^{v(1-\delta)(1-\delta c)} - 1\right)}{e^{v(1-\delta)(1-\delta c)}p_H}$ which I can plug into (26) and get a maximization problem just in terms of δ. Taking the second derivative of this expression and plugging in $(\delta^*)^{MOD}$, I get $-\frac{1}{2}v\left(v + 8\ln\left(\frac{2w_H}{w_H+w_L}\right)\right)$. $2w_H > w_H + w_L$ so this expression is always negative, confirming the solution is a maximum. There were other expressions that fulfilled the first order conditions but they set δ to nonsensical values that were greater than 1, so these solutions were disregarded.

Comparison of Optimal Parameters

I compare δ^* in the original model versus $(\delta^*)^{MOD}$ in the modified model with a negative externality on consumption. Figure 9a plots δ^* and $(\delta^*)^{MOD}$ as a function of the wealth ratio $(\frac{w_H}{w_L})$, holding v fixed and figure 9b plots δ^* and $(\delta^*)^{MOD}$ as a function of the value, holding the wealth ratio fixed.



(a) δ^* and $(\delta^*)^{MOD}$ vs the Wealth Ratio

(b) δ^* and $(\delta^*)^{MOD}$ vs the Value

Figure 9: (a) shows δ^* and $(\delta^*)^{MOD}$ versus the wealth ratio where $x = \frac{w_H}{w_L}$ with v arbitrarily set to 0.2. (b) shows the δ^* and $(\delta^*)^{MOD}$ versus the value where $x = v$ with $\frac{w_H}{w_L}$ arbitrarily set to 3.

As stated above, $(\delta^*)^{MOD} = 1$ when $\frac{w_H}{w_L} = 1$. By assumption, Elon's wealth is greater than Bob's wealth so I always want to ration by setting $(\delta^*)^{MOD} < 1$. In the original model, Elon's wealth had to be at least twice as large as Bob's wealth before $\delta^* < 1$ and Bob's supply of the good is rationed. The conditions for rationing are less strict in the modified case due to this negative externality on consumption. All else equal, if Bob's consumption poses a negative externality on Elon, there should be less consumption of the good and this leads to a smaller value of δ. In the relevant range $0 \leq \delta \leq 1$, $(\delta^*)^{MOD}$ is strictly less than δ^* for all values of $\frac{w_H}{w_L}$ and v.

Looking at the high price, $(p_H^*)^{MOD} \geq p_H^* \quad \forall w_H, w_L$. This holds with equality only when $\delta^* = 0$ and $(\delta^*)^{MOD} = 0$ where the two problems become identical since there is no externality if Bob does not consume the good. There are two opposing effects on the high price when I move from the original model to the modified model. With congestion, Elon's willingness to pay is lower given the same δ because he values the good less compared to the case without congestion, so this should decrease p_H. However, an opposing effect is that holding the initial parameters fixed (w_H, w_L, and v), $(\delta^*)^{MOD}$ is smaller than δ^* so I am able to charge Elon a higher price since the low price is less attractive to him. In this case, it seems that the net effect is to increase the high price relative to the original model.