Study Guide for Analysis

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This study guide was written to help you prepare for the analysis portion of the Comprehensive and Honors Qualifying Examination in Mathematics. It is based on the *Syllabus for Analysis* (*Math 355*) available on the Department website.

Each topic from the syllabus is accompanied by a brief discussion and examples from old exams. When reading this guide, you should focus on three things:

- *Understand the ideas.* If you study problems and solutions without understanding the underlying ideas, you will not be prepared for the exam.
- *Understand the strategy of each proof.* Most proofs in this guide are short—the hardest part is often knowing where to start. Focus on this rather than falling into the trap of memorizing proofs.
- *Understand the value of scratchwork.* Sometimes scratchwork is needed before you start the proof.

The final section of the guide has some further suggestions for how to prepare for the exam.

1 Mathematical Induction

Induction in a powerful method of proof. Be prepared to do a proof by induction. Here is an example.

1 (January 2010) Use induction to prove that

$$\sum_{k=1}^{n} kx^{k-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$$

for all positive integers n.

You should be able to prove this without difficulty. The proof will give you some good algebra practice. Here is a tricker example where scratch work is needed.

2 (February 2006) Prove that for every positive integer n, the following inequality is true:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

Scratchwork. Adding $\frac{1}{\sqrt{n+1}}$ to each side, the right becomes $2\sqrt{n} + \frac{1}{\sqrt{n+1}}$. Is this $< 2\sqrt{n+1}$? Check:

$$2\sqrt{n} + \tfrac{1}{\sqrt{n+1}} < 2\sqrt{n+1} \overset{\text{clear denom}}{\iff} 2\sqrt{n(n+1)} + 1 < 2(n+1) \overset{\text{algebra}}{\iff} \sqrt{n(n+1)} < n + \tfrac{1}{2}.$$

The last inequality is easily checked by squaring each side. Now you can write the proof.

See **12** and **15** for further examples of induction problems.

2 The Real Numbers

The properties of the real numbers underlie everything that happens in analysis.

Rational and Irrational Numbers. Know the following basics:

- The definition of rational number and irrational number.
- Rational numbers are closed under addition, subtraction, multiplication, and division by nonzero rational numbers. In other words, the rational numbers form a field.
- Rational numbers are dense: there is a rational number between any two distinct real numbers. The same is true for irrational numbers. The density of rationals and irrationals is used in **9** and **29**.

Real Numbers and the Axiom of Completeness. The set $\mathbb{R}$ of real numbers has the properties:

- $\mathbb{R}$ is a field (closed under addition, subtraction, multiplication, and division by nonzero real numbers).
- For any $a, b \in \mathbb{R}$, exactly one of $a > b, a = b$, $a < b$ is true. Also, if $a > b$, then $a + c > b + c$ for all $c \in \mathbb{R}$, and $ac > bc$ when $c > 0$. This means that $\mathbb{R}$ is an ordered field.

Given a subset $A \subseteq \mathbb{R}$, know the definitions of upper bound of A and lower bound of A. Definitely know the Axiom of Completeness, which says that if $A \subseteq \mathbb{R}$ is nonempty and bounded above, then it has a least upper bound $\sup(A)$. Be prepared to say more, namely that "$\sup(A)$ is the least upper bound" means

- $\sup(A)$ is an upper bound, i.e., $x \leq \sup(A)$ for all $x \in A$.
- If b is any upper bound of A, then $\sup(A) \leq b$.

Also know that nonempty bounded below set $B \subseteq \mathbb{R}$ has a greatest lower bound $\inf(B)$, and be able to say this precisely as above. Here is a problem that uses both inf and sup.

3 (March 2011) Let A, B be nonempty subsets of the set of real numbers $\mathbb{R}$ such that $x < y$ for every $x \in A$ and $y \in B$. Prove that $\sup(A) \leq \inf(B)$.

Proof. Fix $x \in A$. By hypothesis, x is a lower bound of B, so $\inf(B)$ exists by completeness and $x \leq \inf(B)$ since $\inf(B)$ is the greatest lower bound. Since $x \in A$ was arbitrary, $\inf(B)$ is an upper bound of A, so $\sup(A)$ exists and $\sup(A) \leq \inf(B)$ since $\sup(A)$ is the least upper bound. QED

Comment. Another way to start is "Fix $y \in B$." Can you complete the proof from here?

If a subset $A \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup(A)$ is an upper bound of A and $\forall\, \varepsilon > 0$, $\sup(A) - \varepsilon$ is not an upper bound, so $\exists\, x \in A$ with $\sup(A) - \varepsilon < x$. Know this and be able to draw a picture. Also know the analogous statement and picture for inf. See **5** for a problem that uses this.

3 Sequences

Sequences and their properties are surprisingly important in analysis.

Convergence. A sequence of real numbers is denoted (a_n) (sometimes $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$). Know:

- The ε-N definition of convergence.
- Basic results about sums, products and constant multiples of convergent sequences.
- Be able to prove the result about sums. This is an $\varepsilon/2$ argument, similar to **15**.
- Be able to prove that a convergent sequence is bounded, similar to **6**.

Here is a problem involving sequences and inequalities.

4 (March 2012) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers with the property that $a_n < b_n$ for all n. Use the definition of convergence to prove that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Proof. Let $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$. We will assume $L > M$ and derive a contradiction. Let $\varepsilon = \frac{1}{2}(L - M)$ and note that

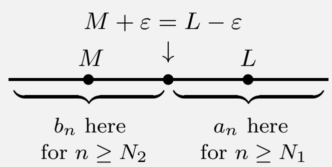
$$M + \varepsilon = L - \varepsilon$$

since ε is half the distance between M and L. Then pick N_1 such that $|a_n - L| < \varepsilon$ for $n \geq N_1$ and N_2 such that $|b_n - M| < \varepsilon$ for $n \geq N_2$. Then for $n \geq \max(N_1, N_2)$, we have

$$L - \varepsilon < a_n < L + \varepsilon \text{ and } M - \varepsilon < b_n < M + \varepsilon, \text{ hence } b_n < M + \varepsilon = L - \varepsilon < a_n,$$

which contradicts $a_n < b_n$. Hence we must have $L \leq M$. QED

Comment. Contradiction is a standard proof strategy, so $L > M$ is a good place to start. To use the definition of convergence, you need to pick ε, which will depend on how close M and L are. This is why $\varepsilon = \frac{1}{2}(L - M)$ is reasonable. Here is a picture showing $L > M$ and ε:



For many students, having a picture is a big help. It is okay to include a picture in your proof.

Caution. Some students overuse contradiction. For most problems, a direct proof is better. Of the 31 problems in this Study Guide, contradiction is used only three times, in **4**, **10** and **11**. This is about 10% of the total and is a good indication of how sparingly contradiction should be used. A good understanding of the ideas often leads quickly to a direct proof. You should look at the various proofs to see why contradiction is rarely needed yet is genuinely useful in some cases.

Convergence of Bounded Monotone Sequences. Know that a bounded above increasing sequence converges and that a bounded below decreasing sequence converges. Some problems require that you use these facts. Also be prepared to prove these facts, which you do using the Axiom of Completeness.

5 (February 2007) Prove that a decreasing sequence of real numbers that is bounded below must converge.

Proof. Let the sequence be (a_n). Then the set $A = \{a_1, a_2, \dots\}$ is nonempty and bounded below, so $L = \inf(A)$ exists by the Axiom of Completeness. To prove $\lim_{n\to\infty} a_n = L$, take $\varepsilon > 0$. Since $L = \inf(A)$, $L + \varepsilon$ is not a lower bound, so $\exists\, a_N \in A$ with $a_N < L + \varepsilon$. Since (a_n) is decreasing, $n \geq N$ implies $a_n \leq a_N < L + \varepsilon$. But we also have $\inf(A) = L \leq a_n$ since $\inf(A)$ is an lower bound. Thus

$$L - \varepsilon < L \leq a_n < L + \varepsilon$$

for $n \geq N$, i.e., $|a_n - L| < \varepsilon$. QED

Comment. Notice how the proof distinguishes between the sequence (a_n) and the set $A = \{a_1, a_2, \dots\}$. These are not the same. This is the kind of careful thinking you want to bring to the exam.

Cauchy Sequences. Know:

- The definition of Cauchy sequence.

- Every Cauchy sequence converges.
- Be able to prove that a convergent sequence is Cauchy (see **6**).
- Be able to prove that a sum of Cauchy sequences is Cauchy (another $\varepsilon/2$ argument.)

6 (January 2016) Let (a_n) be a convergent sequence of real numbers. Prove that (a_n) is a Cauchy sequence.

Proof. This proof uses the triangle inequality. Let $a = \lim_{n\to\infty} a_n$ and pick an arbitrary $\varepsilon > 0$. Because (a_n) converges, $\exists\, N \in \mathbb{N}$ such that if $k \geq N$, then $|a_k - a| < \varepsilon/2$. Suppose $m, n \geq N$. Then

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| = |a_n - a| + |a_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Because ε is arbitrary, (a_n) is Cauchy. QED

Here is a problem involved boundedness.

7 (January 2010) Use the definition of Cauchy sequence to prove that every Cauchy sequence in $\mathbb{R}$ is bounded.

Proof. Suppose (a_n) is a Cauchy sequence. By definition, for $\varepsilon = 1$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_m - a_n| < 1$. For $n = N$, we obtain $|a_m - a_N| < 1$ for all $m \geq N$. By the triangle inequality, this implies

$$|a_m| - |a_N| \leq |a_m - a_N| < 1$$

for $m \geq N$. Thus $|a_m| < |a_N| + 1$ for $m \geq N$. Let $M = \max(|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1)$. Then $|a_k| \leq M$ for all $k \in \mathbb{N}$. By definition, the sequence is bounded by M. QED

Comment. Notice that you use the definition only for one value of ε (namely $\varepsilon = 1$) to show that $|a_m|$ is bounded for m large. The "max" takes care of the remaining terms of the sequence.

The standard version of the triangle inequality says $|a + b| \leq |a| + |b|$. However, the proof in **7** uses the version that says $|a| - |b| \leq |a - b|$. Know both. Inequalities lie at the heart of analysis.

Bolzano-Weierstrass Theorem for Sequences. Know the version of the Bolzano-Weierstrass Theorem that applies to bounded sequences of real numbers.

4 Point-Set Theory

Analysis has developed a rich language for describing subsets of $\mathbb{R}$.

Limit Points. Given $A \subseteq \mathbb{R}$, know:

- The definition of a limit point a of A in terms of ε-neighborhoods $V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$.
- The theorem that a is a limit point of A if and only if $a = \lim a_n$ for some sequence (a_n) with $a_n \in A$ and $a_n \neq a$ for all n.

Be prepared to prove the theorem just mentioned.

8 (March 2014) A real number x is called a *limit point* of a set S of real numbers if every open interval containing x also contains a point of S distinct from x. Use this definition to prove that x is an limit point of S if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points in $S \setminus \{x\}$ (set S with the point x removed) such that $x = \lim_{n\to\infty} x_n$.

Proof. First assume that x is a limit point of S. For $n \geq 0$, we can pick $x_n \in (x - 1/n, x + 1/n)$ with $x_n \in S$ and $x_n \neq x$. Thus

$$x - \tfrac{1}{n} < x_n < x - \tfrac{1}{n}, \text{ so } |x_n - x| < \tfrac{1}{n}.$$

To prove that $x = \lim_{n\to\infty} x_n$, take $\varepsilon > 0$ and pick $N > 1/\varepsilon$. If $n \geq N$, then

$$|x_n - x| < \tfrac{1}{n} < \tfrac{1}{N} < \varepsilon,$$

proving that $\{x_n\}_{n=1}^{\infty}$ converges to x.

Now assume $x = \lim_{n\to\infty} x_n$ where $x_n \in S \setminus \{x\}$ for all n. If I is an open interval containing x, we can find $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq I$. Since $\{x_n\}_{n=1}^{\infty}$ converges to x, there is N such that $n \geq N$ implies $|x_n - x| < \varepsilon$. Then in particular $|x_N - x| < \varepsilon$, so $x_N \in (x - \varepsilon, x + \varepsilon) \subseteq I$. Since $x_N \in S$ and $x_N \neq x$, I contains a point of S different from x, so x is a limit point of S. QED

Comment. Be sure you understand where $N > 1/\varepsilon$ comes from in the first part of the proof.

Also know how to find the set all limit points of a given set.

9 (March 2010) Find the limit points of the following sets:

(a) The rationals.
(b) The irrationals.
(c) The interval $[a, b)$ for $a < b$.
(d) The integers.

Solution. (a) Since the rationals are dense in $\mathbb{R}$, every interval contains infinitely many rational numbers. Hence the set of limit points is all of $\mathbb{R}$.

(b) Since the irrationals are dense in $\mathbb{R}$, similar reasoning shows that the set of limit points is $\mathbb{R}$.

(c) The answer is clearly the closed interval $[a, b]$.

(d) The answer is $\emptyset$ since $a \notin \mathbb{Z}$ lies in an interval disjoint from $\mathbb{Z}$ and $a \in \mathbb{Z}$ lies in an interval whose only point of $\mathbb{Z}$ is a itself.

Comment. The problem only asks for the answers. But the reasons can help with partial credit. Also, if the problem had said "Justify your answers", then the reasons would be essential.

Limit points are sometimes called cluster points or accumulation points.

Bolzano-Weierstrass Theorem for Sets. Know the version of the Bolzano-Weierstrass Theorem that applies to bounded infinite subsets of $\mathbb{R}$.

Open and Closed Sets. Know the definition of open and closed subset of $\mathbb{R}$.

10 (March 2009) The boundary ∂A of a set A of real numbers is the set of points $x \in \mathbb{R}$ such that for every $\varepsilon > 0$, the ε-neighborhood $V_\varepsilon(x) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$ contains a point in A and a point not in A.

Prove that A is closed (i.e., A contains all of its limit points) if and only if it contains its boundary.

Proof. Assume that A is closed and let $x \in \partial A$. We want to prove $x \in A$. We will assume $x \notin A$ and derive a contradiction. Since $x \in \partial A$, every ε-neighborhood of x contains points in A and not in A. The point in A must be different from $x \notin A$, so every ε-neighborhood of x contains a point in A different from x. Thus x is a limit point of A hence $x \in A$ since A is closed. This contradicts $x \notin A$.

Assume $\partial A \subseteq A$ and let x be a limit point of A. If we can prove $x \in A$, then A contains its limit points and hence is closed. We will assume $x \notin A$ and derive a contradiction. Since x is a limit point of A, every ε-neighborhood of x contains a point in A, and it also contains a point not in A, namely, $x \notin A$. Hence $x \in \partial A \subseteq A$, which contradicts $x \notin A$. QED

Comment. This problem may look confusing if you have never seen the concept of boundary. If you get stuck, you can get partial credit by stating the first two sentences of each part of the proof. This problem is an example where contradiction is a good strategy. To succeed in the problem, you need to be able to read a new definition (here, the definition of boundary) and act on it.

Compact Sets and the Heine-Borel Theorem. There are various ways to say what it means for a subset $K \subseteq \mathbb{R}$ to compact. The Heine-Borel Theorem states the the following are equivalent:

- Every sequence in K has a subsequence that converges to a point of K.
- K is closed and bounded.
- Every open cover of K has a finite subcover.

Any of these conditions can serve as the definition of compact. Here is a problem that uses these ideas.

11 (February 2006) Recall that a compact set C has the property that every open cover has a finite subcover. Use this characterization of compactness to show the interval $[0, \infty)$ of real numbers is not compact.

Proof. Let $O_n = (-1, n)$ for $n \geq 1$. It is clear that $\bigcup_{n=1}^{\infty} O_n = (-1, \infty)$, so $[0, \infty) \subseteq \bigcup_{n=1}^{\infty} O_n$. Hence $\{O_n\}_{n=1}^{\infty}$ is an open cover of $[0, \infty)$. Suppose there is a finite subcover

$$[0, \infty) \subseteq O_{n_1} \cup \cdots \cup O_{n_s}.$$

If $N = \max(n_1, \ldots, n_s)$, then we have a problem since $N + 1 \notin O_{n_1} \cup \cdots \cup O_{n_s}$ yet $N + 1 \in [0, \infty)$. This contradiction shows that no finite subcover exists, hence $[0, \infty)$ is not compact. QED

5 Infinite Series

Infinite series have been part of analysis for over 350 years. Many important concepts in analysis first arose in the context of infinite series.

Convergence. Know the definition of convergence of an infinite series as the limit of its partial sums.

12 (February 2007) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

(a) Find a simple formula (not involving summation notation) for the kth partial sum

$$\sum_{n=1}^{k} \frac{1}{n(n+1)},$$

and prove that it is correct.
(b) Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges (and state clearly what it conveges to).

Proof. (a) To find the formula, the idea is to begin with some scratchwork:

$$k = 1\colon \frac{1}{1 \cdot 2} = \frac{1}{2}$$
$$k = 2\colon \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{3+1}{6} = \frac{4}{6} = \frac{2}{3}$$
$$k = 3\colon \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{6+2+1}{12} = \frac{9}{12} = \frac{3}{4}.$$

This suggests that $\sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{k}{k+1}$. You should be able to prove this by induction.
(b) By definition, the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is the limit of its partial sums:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n(n+1)} = \lim_{k\to\infty} \frac{k}{k+1} = 1.$$

This proves that the series converges with sum equal to 1. QED

Convergence Criteria. You studied infinite series in calculus and more rigorously in analysis. Here are some things you are expected to know:

- Convergence criteria for p-series and geometric series.
- The concepts of absolute and conditional convergence and the relation between them.
- The Comparison, Ratio, and Alternating Series Tests.

For example, the Weierstrass M-test problem **27** requires knowing when a geometric series converges, and the power series problem **28** uses the Ratio Test to determine the radius of convergence and the p-series and the Alternating Series Test to analyze the behavior at the endpoints of the interval of convergence.

6 Limits and Continuity

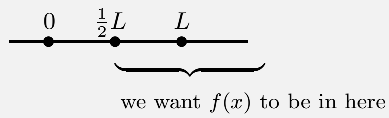
Limits and continuity are studied in calculus, but proving their properties requires the full power of analysis.

The Limit of a Function. Given a function $f : A \to \mathbb{R}$ for $A \subseteq \mathbb{R}$ and a limit point c of A, know the ε-δ definition of $\lim_{x\to c} f(x) = L$. Know how to use this definition, as in the following example.

13 (January 2011) Assume f is defined on $\mathbb{R}$ and $\lim_{x\to 0} f(x) = L$ for some real number $L > 0$. Prove that there is $\delta > 0$ with the property that $f(x) > \frac{1}{2}L$ for all $x \in (-\delta, \delta) \setminus \{0\}$.

Proof. Let $\varepsilon = \frac{1}{2}L$. By the definition of limit, we know there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$, we have $|f(x) - L| < \varepsilon = \frac{1}{2}L$. For this δ, we get $\frac{1}{2}L < f(x) < \frac{3}{2}L$ for all $x \in (-\delta, \delta) \setminus \{0\}$. In particular, we have $f(x) > \frac{1}{2}L$ for $x \in (-\delta, \delta) \setminus \{0\}$. This is what we wanted to prove. QED

Comment. The key step in the proof is picking $\varepsilon = \frac{1}{2}L$. If you draw a picture, this is easy to see:



What ε would you have chosen if the problem had asked for $f(x) > .9L$?

You should also know:

- Basic facts about sums, product and quotients of limits.
- Limits preserve inequalities, i.e., if $f(x) \geq g(x)$ for all $x \in A$, then $\lim_{x \to c} f(x) \geq \lim_{x \to c} g(x)$ when both limits exist.

Be prepared to prove the facts about sums and inequalities. For sums, the proof is similar to **15**, while for inequalities, the proof is similar to **4**. You should do these proofs for practice.

The Definition of Continuity. Given $f : A \to \mathbb{R}$ for $A \subseteq \mathbb{R}$ and $c \in A$, know:

- The ε-δ definition of f being continuous at c.
- The characterization of continuity of f at c in terms of sequences $(x_n) \to c$.

Be prepared to prove the sequence characterization of continuity.

14 (January 2014) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at c and $\{x_n\}$ is a sequence of real numbers that converges to c. Use the ε-δ definition of continuity to prove that the sequence $\{f(x_n)\}$ converges to $f(c)$.

Proof. Let $\varepsilon > 0$. Since f is continuous at c, there is $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$, and since $\{x_n\}$ converges to c, there exists N such that $n \geq N$ implies $|x_n - c| < \delta$. Thus

$$n \geq N \implies |x_n - c| < \delta \implies |f(x_n) - f(c)| < \varepsilon,$$

proving that $\{f(x_n)\}$ converges to $f(c)$. QED

Comment. There are two hypotheses: f continuous at c and $\{x_n\}$ converges to c. Which do you use first? Since you want to end with $|f(x_n) - f(c)| < \varepsilon$, the answer is f. This is another example of how scratchwork can help you begin the proof.

Also know what it means for a function $f : A \to \mathbb{R}$ to be continuous on A.

Continuity of Sums, Products, Quotients and Compositions. Know careful statements of these results and be prepared to supply proofs for sums and compostions.

15 (March 2012) (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Give a careful ε-δ proof that the sum function $f + g : \mathbb{R} \to \mathbb{R}$ defined by $(f + g)(x) = f(x) + g(x)$ is also continuous.

(b) Now suppose that we have continuous functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$, $n \geq 2$, and define $f_1 + \cdots + f_n$:

$\mathbb{R} \to \mathbb{R}$ by $(f_1 + \cdots + f_n)(x) = f_1(x) + \cdots + f_n(x)$. Use part (a) and induction to prove that $f_1 + \cdots + f_n$ is continuous.

Proof. (a) (A classic $\varepsilon/2$ argument) Given $\varepsilon > 0$ and $c \in \mathbb{R}$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|x - c| < \delta_1 \;\Rightarrow\; |f(x) - f(c)| < \tfrac{1}{2}\varepsilon \text{ and } |x - c| < \delta_2 \;\Rightarrow\; |g(x) - f(c)| < \tfrac{1}{2}\varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$ and assume $|x - c| < \delta$. Then $|x - c| < \delta_i$ for $i = 1, 2$, so by the definition of $f + g$ and the triangle inequality, we obtain

$$\begin{aligned} |(f+g)(x) - (f+g)(c)| &= |f(x) + g(x) - (f(c) + g(c))| \\ &= |(f(x) - f(c)) + (g(x) - g(c))| \\ &\le |f(x) - f(c)| + |g(x) - g(c)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

(b) You should prove this. The key observation is that $f_1 + \cdots + f_{n+1} = (f_1 + \cdots + f_n) + f_{n+1}$. QED

Here is a problem involving continuity and composition.

16 (January 2016) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that g is continuous at c and f is continuous at $g(c)$. Use the the ε-δ of continuity to prove that the composite function $f \circ g$, defined by

$$(f \circ g)(x) = f(g(x)),$$

is continuous at c.

Proof. Pick an arbitrary $\varepsilon > 0$. Since f is continuous at $g(c)$, by definition $\exists\, \delta_1 > 0$ such that $\forall\, y \in \mathbb{R}$, $|y - g(c)| < \delta_1 \Rightarrow |f(y) - f(g(c))| < \varepsilon$. Similarly, since g is continuous at c, $\exists\, \delta > 0$ such that $\forall\, x \in \mathbb{R}$, $|x - c| < \delta \Rightarrow |g(x) - g(c)| < \delta_1$. Combining these two inequalities, we have

$$|x - c| < \delta \Rightarrow |g(x) - g(c)| < \delta_1 \Rightarrow |f(g(x)) - f(g(c))| < \varepsilon,$$

which is exactly what we need to show $f \circ g$ is continuous at c. QED

Comment. The challenge is to get the order correct. You want to end up with

$$|x - c| < \delta \Rightarrow |f(g(x)) - f(g(c))| < \varepsilon,$$

which means that your scratchwork begins with $|f(g(x)) - f(g(c))| < \varepsilon$. The key thing is realize that this is where you use the continuity of f and where δ_1 enters the picture.

There is an interesting family resemblance between **16** and the earlier problem **14**:

- **16**: This can be stated as $\lim_{x \to c} f(g(x)) = f(g(c))$. The strategy of the proof is to start with $|f(g(x)) - f(g(c))| < \varepsilon$, use f continuous at $g(c)$ to find δ_1, and then use δ_1 and g continuous at c to find the desired δ.
- **14**: This can be stated as $\lim_{n \to \infty} f(x_n) = f(c)$ when (x_n) converges to c. The strategy is to start with $|f(x_n) - f(c)| < \varepsilon$, use f continuous at c to find δ, and then use δ and $\lim_{n \to \infty} x_n = c$ to find the desired N.

The Intermediate Value Theorem. Know the precise statement of the Intermediate Value Theorem. Here is an example of how to use it.

17 (February 2007) Suppose that $f : [-1, 1] \to \mathbb{R}$ is continuous and satisfies $f(-1) = f(1)$. Use the Intermediate Value Theorem to prove that there exists $\gamma \in [0, 1]$ such that $f(\gamma) = f(\gamma - 1)$.

Proof. Let $g(x) = f(x) - f(x - 1)$. Since f is continuous and defined on $[-1, 1]$, it follows that g is continuous and defined on $[0, 1]$. Furthermore,

$$\begin{aligned} g(0) &= f(0) - f(-1) = f(0) - f(1) \\ g(1) &= f(1) - f(0), \end{aligned}$$

where the first line uses $f(-1) = f(1)$. It follows that $g(0) = -g(1)$. There are now three cases:

Case 1: $g(0) = 0$. Then $f(0) - f(1) = 0$. Since $f(-1) = f(1)$, $f(\gamma) = f(\gamma - 1)$ is true for $\gamma = 0$.

Case 2: $g(0) > 0$. Then $g(1) < 0$, so $g(0) > 0 > g(1)$. Since g is continuous on $[0, 1]$, the Intermediate Value Theorem implies that there is $c \in (0, 1)$ with $0 = g(c) = f(c) - f(c - 1)$, so $f(\gamma) = f(\gamma - 1)$ holds with $\gamma = c$.

Case 3: $g(0) < 0$. Then $g(1) > 0$, so $g(0) < 0 < g(1)$. Then, as in Case 2, the Intermediate Value Theorem gives the desired result. QED

Comment. Where did $g(x) = f(x) - f(x - 1)$ come from? The nicest version of the Intermediate Value Theorem says that if F is continuous on $[a, b]$ and is positive at one endpoint and negative at the other, then F equals zero somewhere inbetween, i.e., $\exists\, c \in (a, b)$ with $F(c) = 0$. Since the above problem looks strange, recasting it as a function $= 0$ is a good place to start. Since $f(\gamma) = f(\gamma - 1)$ is equivalent to $f(\gamma) - f(\gamma - 1) = 0$, it now makes sense to use $g(x) = f(x) - f(x - 1)$.

Continuous functions on a compact set have three nice properties. Here are the first two.

Boundedness. Know that a continuous function on a compact set is bounded.

Extreme Values. The Extreme Value Theorem says that if $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then there exist $x_0, x_1 \in K$ such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in K$.

18 (March 2010) Let $f : E \to \mathbb{R}$ be a continuous function, where E is a compact set of real numbers. Suppose that $f(x) > 0$ for all $x \in E$. Prove that there exists $M > 0$ such that $f(x) > M$ for all $x \in E$.

Proof. By the Extreme Value Theorem, $\exists\, x_0 \in E$ with $f(x) \ge f(x_0)\ \forall\, x \in E$. By hypothesis, $f(x_0) > 0$ since $x_0 \in E$. Thus $M = \frac{1}{2} f(x_0)$ satisfies $M > 0$ and $f(x) > f(x_0) > M\ \forall\, x \in E$. QED

Comment. The key point is recognizing the role of the Extreme Value Theorem. The hypothesis that f is continuous on a compact set should signal that the Extreme Value Theorem may be relevant.

Uniform Continuity. When $f : A \to \mathbb{R}$ is continuous, then f is continuous at all $c \in A$, which means

$$\forall\, c \in A\, \forall\, \varepsilon > 0\, \exists\, \delta > 0 \text{ such that if } x \in A \text{ and } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$$

This allows δ to depend on both ε and c. When $f : A \to \mathbb{R}$ is uniformly continuous, we instead have

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 \text{ such that if } x, c \in A \text{ and } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$$

In uniform continuity, δ depends only on ε. You should be able to do the following problem.

19 (March 2013) Suppose that $f, g : A \to \mathbb{R}$ are uniformly continuous functions. Prove that the function $f + g : A \to \mathbb{R}$ given by

$$(f + g)(x) = f(x) + g(x)$$

is uniformly continuous.

An important result is that a continuous function on a compact set is automatically uniformly continuous. This is the third nice property of continuous functions on compact sets. The proof uses open cover formulation of compactness from the Heine-Borel Theorem and is one of main applications of Heine-Borel.

7 Differentiability and Derivatives

Derivatives are defined in calculus, but ideas from analysis are needed for the proofs.

The Limit Definition of Derivative. Know the limit definition of the derivative $f'(c)$ of a function f at a point c. There are actually two ways to write the limit definition. Know both (see **20** for one of them).

Derivatives at Local Extreme Points. Given a function f, know what it means for $f(c)$ to be a local maximum value or a local minimum value. Here, c is called a local extreme point. Also know that $f'(c) = 0$ when f is differentiable at a local extreme point c. Be prepared to prove this.

20 (January 2016) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f has a local maximum at c. Prove that $f'(c) = 0$.

Proof. Because $f(c)$ is a local maximum value of f, we may choose $a, b \in \mathbb{R}$ such that $a < c < b$ and $f(x) \le f(c)\ \forall x \in (a, b)$. By definition,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

For every $x \in (a, c)$, we have $f(x) - f(c) \le 0$ and $x - c < 0$, so $\frac{f(x)-f(c)}{x-c} \ge 0$, which means that $f'(c) \ge 0$. Similarly, for every $y \in (c, b)$, $f(y) - f(c) \le 0$ but $y - c > 0$, so $f'(c) \le 0$. Combining these two inequalities, we have $0 \le f'(c) \le 0$, so $f'(c) = 0$, as desired. QED

Comment. How do you know to write $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$? The answer lies in scratchwork, where you would start by noting that $f(x) \le f(c)$ when x is close to c. This gives $f(x) - f(c) \le 0$, which suggests using a definition of $f'(c)$ that involves $f(x) - f(c)$.

The Mean Value Theorem. Know a careful statement of the Mean Value Theorem and how to use it.

21 (January 2015) Suppose a function f has the following properties:

- the domain of f is $\mathbb{R}$;
- f is differentiable at every real number;
- $f'(x) = 0$ for every real number x.

Prove that there is a constant K such that $f(x) = K$ for every real number x. (Hint: Show that $f(x) = f(0)$ for all x.)

Proof. Fix an arbitrary $x \in \mathbb{R} \setminus \{0\}$. First assume $x > 0$. Then f is differentiable on $(0, x)$ and, because it is differentiable on $\mathbb{R}$, it is continuous on $[0, x]$. By the Mean Value Theorem, we may choose $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$

But by assumption, $f'(c) = 0$, and because $x \neq 0$, it must be true that $f(x) - f(0) = 0$. Next assume $x < 0$. Repeating the argument just given with $[x, 0]$ in place of $[0, x]$ shows that $f(0) - f(x) = 0$. Since $f(x) = f(0)$ is obvious when $x = 0$, we conclude that

$$f(x) = f(0) \; \forall \, x \in \mathbb{R}.$$

Thus $f(0)$ is the desired constant K. QED

It is also helpful to know Rolle's Theorem, which is the special case of the Mean Value Theorem when $f(a) = f(b)$. Rolle's Theorem has an especially simple conclusion, namely $f'(c) = 0$.

8 Sequences of Functions

Given the importance of sequences and functions, it makes sense that sequences of functions are a big deal.

Pointwise and Uniform Convergence. A sequence of functions defined on $A \subseteq \mathbb{R}$ is usually denoted (f_n), though sometimes you will see $\{f_n\}$ or $\{f_n\}_{n=1}^{\infty}$. Given a sequence (f_n) defined on A, there are two basic definitions to know:

- (f_n) converges pointwise to f if for every $x \in A$, the sequence of numbers $(f_n(x))$ converges to the number $f(x)$. In other words,

$$\forall \, x \in A \, \forall \, \varepsilon > 0 \, \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N.$$

- (f_n) converges uniformly to f if

$$\forall \, \varepsilon > 0 \, \exists N \in \mathbb{N} \text{ such that } \forall \, x \in A, \; |f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N.$$

The only difference is the location of the quantifier $\forall \, x \in A$, but the impact can be dramatic: in pointwise convergence, N may depend on both x and ε, while in uniform convergence, N depends *only* on ε and works for *all* $x \in A$. Here are two examples of how to use the definition.

22 (March 2006) Let $\{f_n\}$ be a sequence of bounded functions on $[a, b]$. Prove that if $\{f_n\}$ converges uniformly to f on $[a, b]$, then f is bounded.

Proof. Setting $\varepsilon = 1$ in the definition of uniform convergence gives $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < 1$ for all $x \in [a, b]$ and all $n \geq N$. In particular, $|f(x) - f_N(x)| = |f_N(x) - f(x)| < 1$ for all $x \in [a, b]$. By the triangle inequality,

$$|f(x)| - |f_N(x)| < 1 \quad \Longrightarrow \quad |f(x)| < |f_N(x)| + 1$$

for $x \in [a, b]$. Since f_N is bounded, there exists M such that $|f_N(x)| \leq M$ for all $x \in [a, b]$. Thus

$$|f(x)| < |f_N(x)| + 1 \leq M + 1$$

for $x \in [a, b]$, proving that f is bounded on $[a, b]$.

23 (March 2016) Assume that $\{f_n\}$ converges uniformly to f on A and that f is bounded on A. Prove that there is an integer N such that f_n is bounded on A for every $n \geq N$.

Proof. Setting $\varepsilon = 1$ in the definition of uniform convergence gives $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < 1$ for all $x \in A$ and all $n \geq N$. By the triangle inequality,

$$|f_n(x)| - |f(x)| < 1 \implies |f_n(x)| < |f(x)| + 1$$

for $x \in A$ and $n \geq N$. Since f is bounded, there exists M such that $|f(x)| \leq M$ for all $x \in A$. Thus

$$|f_n(x)| < |f(x)| + 1 \leq M + 1$$

for $x \in A$ and $n \geq N$, proving that f_n is bounded on A when $n \geq N$. QED

It is instructive to compare **6**, **22** and **23**. They all use $\varepsilon = 1$ and $|a| - |b| \leq |a - b|$, but there are some important differences:

- **22**: The f_n are bounded and we want to prove that the limit is bounded. So only f_N is needed.
- **23**: We instead assume the limit is bounded and want to prove that f_n is bounded for n large.
- **6**: (a_n) is a sequence of numbers. Using $\varepsilon = 1$ gives a bound on $|a_n|$ for n large, but since we want to bound the whole sequence, we use max to handle the remaining terms.

Comparing these proofs will give you a better sense of how basic ideas can lead to proofs that are similar in spirit but differ in the details. We have seen students memorize proofs such as **6** and then apply them rigidly to situations like **22** and **23** where max is not relevant.

Continuity of the Limit. Know that if each f_n is continuous on A and (f_n) converges uniformly to f, then f is continuous. Sometimes this can be used to give a quick proof that uniform convergence fails.

24 (March 2008) Let $f_n(x) = \frac{1}{1+n^2x^2}$. Prove that $\{f_n\}_{n=1}^{\infty}$ converges pointwise but not uniformly on the interval $[0, 1]$.

Proof. The key insight is that for $x \in [0, 1]$, the denominator $1 + n^2x^2$ gets large as $n \to \infty$ except when $x = 0$. More precisely, $f_n(0) = \frac{1}{1+n^2 \cdot 0^2} = 1$ for all n, while for $x \in (0, 1]$, $1 + n^2x^2 \to \infty$ as $n \to \infty$ since $x > 0$. It follows that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0, 1]. \end{cases}$$

This shows that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a discontinous function. Since each f_n is continuous (standard formulas from calculus show that f_n is differentiable, hence continuous). Since a uniform limit of continuous functions is continuous, $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly. QED

Proving Uniform Convergence. Because the limit was discontinous, **24** ended up being easy. When the limit is continuous, problems involving uniform convergence can require more work. We present two examples.

25 (January 2014) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $n \in \mathbb{N}$.

(a) State the function f to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise.

(b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1, \infty)$.

Proof. (a) When $x = 0$, $f_n(0) = 0$, and when $x \neq 0$, then

$$f_n(x) = \frac{nx}{1 + n^2x^2} = \frac{x}{1/n + nx^2} \to 0$$

as $n \to \infty$ since the numerator is the constant x and $x^2 > 0$ implies that the denominator $\to \infty$ as $n \to \infty$. Thus the limit function is the zero function $f(x) = 0$ for all x.
(b) Given an arbitrary $\varepsilon > 0$, fix $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Fix an arbitrary $x \in [1, \infty)$ and $n \geq N$. Then

$$|f_n(x) - f(x)| = \left|\frac{nx}{1 + n^2x^2} - 0\right| = \frac{nx}{1 + n^2x^2} \leq \frac{nx}{n^2x^2} = \frac{1}{nx} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

where we used $x \geq 1$ and $n \geq N$. Because $x \geq 1$ and $n \geq N$ are arbitrary, it follows that $\{f_n\} \to f$ uniformly on $[1, \infty)$. QED

Comment. How do you know to pick $N > 1/\varepsilon$ in part (b)? The answer lies in scratchwork:

$$|f_n(x) - f(x)| = \left|\frac{nx}{1 + n^2x^2} - 0\right| = \frac{nx}{1 + n^2x^2} \leq \frac{nx}{n^2x^2} = \frac{1}{nx}.$$

This is good time to check the hypothesis, which tells you that $x \in [1, \infty)$. Thus $x \geq 1$, which when combined with the above scratchwork gives

$$|f_n(x) - f(x)| \leq \frac{1}{n}.$$

Since you want this to be $< \varepsilon$ for all $n \geq N$, you know how to pick N. Now you can write the proof.

Here is a harder problem.

26 (March 2009) Show that the sequence of functions $\{f_n\}$ with $f_n(x) = x^n(1 - x)$ converges uniformly on $[0, 1]$ to the zero function.

Proof. Since $0 \leq x < 1 \Rightarrow x^n \to 0$ as $n \to \infty$ and $0 \leq f_n(x) = x^n(1 - x) \leq x^n$, we see that $f_n(x) \to 0$ as $n \to \infty$ when $0 \leq x < 1$. When $x = 1$, $f_n(1) = 1^n(1 - 1) = 0$. It follows that

$$\lim_{n\to\infty} f_n(x) = 0$$

for all $x \in [0, 1]$, so $\{f_n\}$ converges pointwise to the zero function. To prove that the convergence is uniform, we need to find a bound on $|f_n(x) - 0| = f_n(x)$ that goes to zero.

The idea is to find the maximum value of $f_n(x) = x^n(1 - x) = x^n - x^{n+1}$ on $[0, 1]$, which exists by the Extreme Value Theorem. By **20**, it occurs where $f_n'(x) = nx^{n-1} - (n + 1)x^n = 0$, so

$$0 = nx^{n-1} - (n + 1)x^n = x^{n-1}(n - (n + 1)x).$$

Since $f_n(x)$ vanishes at the endpoints $0, 1$, the maximum occurs at $\frac{n}{n+1}$. So for $x \in [0, 1]$, we have

$$f_n(x) \leq \left(\frac{n}{n + 1}\right)^n \left(1 - \frac{n}{n + 1}\right) = \left(\frac{n}{n + 1}\right)^n \frac{1}{n + 1} \leq \frac{1}{n + 1}.$$

You should complete the ε-N proof that $\{f_n\}$ converges uniformly on $[0, 1]$ to the zero function. QED

When you first read **25** and **26**, it is not obvious why one is harder than the other. The first step in each case is to find the limit function $f(x)$. Once you look at $|f_n(x) - f(x)|$, the difference emerges:

- In **25**, $|f_n(x) - f(x)|$ is the fraction $\frac{nx}{1+n^2x^2}$, where the denominator is clearly bigger as n increases. After some scratchwork, finding the proof in **25** is relatively straightforward.
- In **26**, $|f_n(x) - f(x)|$ equals $x^n(1-x)$, which does not have an obvious bound. This is why you need to find the maximum value of $x^n(1-x)$.

9 Series of Functions

Functions represented by infinite series are extremely important in mathematics. Examples include power series and Fourier series. In analysis, one develops a theory that applies to any infinite sum $\sum_{n=1}^{\infty} f_n(x)$.

Pointwise Convergence. If f_n is defined on A for all n, then $\sum_{n=1}^{\infty} f_n$ converges pointwise on A if for all $x \in A$, the series of numbers $\sum_{n=1}^{\infty} f_n(x)$ converges to a number $f(x)$. We write $f = \sum_{n=1}^{\infty} f_n$ in this case.

Uniform Convergence. Assume $\sum_{n=1}^{\infty} f_n$ converges pointwise on A to a function f. There are two ways to define uniform convergence, both of which use the sequence of partial sums:

- $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if the sequence of functions $s_n(x) = \sum_{k=1}^{n} f_k$ converges uniformly on A. However, to give a complete definition, you need to also define uniform convergence for a sequence of functions.
- $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to f if

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \text{ such that } \forall x \in A, |f_1(x) + \cdots + f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N.$$

If each f_n is continuous on A and $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on A, then f is continuous on A. You should be able to prove this using the corresponding result for uniform convergence of sequences.

The Weierstrass M-Test. This is a quick and powerful way to show uniform convergence of a series. Assume f_n is defined on $A \subseteq \mathbb{R}$ and bounded by $|f_n(x)| \leq M_n$ for $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A. Here is an example of how to use this result.

27 (January 2016) Prove that the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges uniformly on the interval $[-r, r]$ for any $0 < r < 1$. (You may use the Weierstrass M-Test if you state the test precisely and explain clearly how it applies.)

Proof. First, let us state the Weierstrass M-Test. Let $f_n : A \to \mathbb{R}$ be a function defined on A $\forall n \in \mathbb{N}$ and let $M_n > 0$ be a real number such that $\forall x \in A$, $|f_n(x)| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.
Fix an arbitrary $r \in (0, 1)$. Let $x \in [-r, r]$. Then $\forall n \in \mathbb{N}$,

$$\left|\frac{x^n}{n}\right| = \frac{|x^n|}{|n|} = \frac{|x^n|}{n} = \frac{|x|^n}{n} \leq \frac{r^n}{n} \leq r^n.$$

(Note that this is a fixed number for each n since r is fixed.) Since $0 < r < 1$, so the geometric series $\sum_{n=1}^{\infty} r^n$ converges. Therefore, by the Weierstrass M-Test with $M_n = r^n$, $\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-r, r]$. Finally, because r is arbitrary, $\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-r, r]$ for all $r \in (0, 1)$.

QED

Power Series. A power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has a radius of convergence R satisfying

$$\sum_{n=0}^{\infty} a_n(x-a)^n \text{ converges absolutely when } |x-a| < R \text{ and diverges when } |x-a| > R.$$

Recall that $R = 0$ and $R = \infty$ are both possible. Also know how to find R using the Ratio Test.

The Interval of Convergence. If the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-a)^n$ satisfies $0 < R < \infty$, then you cannot predict in advance what happens when $|x-a| = R$. These are the endpoints of the interval $(a-R, a+R)$. Adding those endpoints where the series converges (if any) to $(a-R, a+R)$ gives the interval of convergence. Here is a typical problem where you need to analyze the behavior at the endpoints.

28 (January 2014) Find all values x for which the following series converges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n2^n}$$

Solution. If $a_n \neq 0$ for all n, the Ratio Test states that $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$ and diverges if $\lim_{n\to\infty} |a_{n+1}/a_n| > 1$. Applying the test to the above series, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n(x-1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n(x-1)}{2(n+1)}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right|\left|\frac{x-1}{2}\right| = \frac{|x-1|}{2}.\end{aligned}$$

Therefore, the series converges for x satisfying

$$\frac{|x-1|}{2} < 1 \iff |x-1| < 2 \iff -1 < x < 3.$$

Similarly it diverges when $\frac{|x-1|}{2} > 1$, i.e., $x < -1$ or $x > 3$. It remains to study the behavior at the endpoints, where the Ratio Test is inconclusive. When $x = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-1-1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series, which is known to diverge. (Another proof is that this is a p-series with $p = 1$, so it diverges since a p-series converges $\iff p > 1$). When $x = 3$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n(3-1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

The terms of this series alternate and their absolute values decrease $|(-1)^{n+1}/(n+1)| \leq |(-1)^n/n|$ with limit zero. By the Alternating Series Test, the series converges. Hence the given power series converges for $x \in (-1, 3]$.

Continuity and Differentiation of Power Series. A power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence $R > 0$ is differentiable (hence continuous) on $(a-R, a+R)$, and its derivative is given by

$$\Big(\sum_{n=0}^{\infty} a_n(x-a)^n\Big)' = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}.$$

Also, R is the radius of convergence of the power series $\sum_{n=1}^{\infty} na_n(x-a)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-a)^n$.

10 Integration

Integrals are an important part of calculus, but only in analysis do you see a careful definition of the integral and a proof that the integral of a continuous function actually exists.

Definition of the Riemann integral. Given a bounded function $f : [a,b] \to \mathbb{R}$, know:

- The definition of a partition P of $[a,b]$.
- The definition of the upper sum $U(f,P)$ and the lower sum $U(f,P)$.

A useful result to know is that if P and Q are any partitions of $[a,b]$, then

$$L(f,P) \le U(f,Q),$$

i.e., any lower sum is less than or equal to any upper sum. Using **3**, it follows that

$$\sup\{L(f,P) \mid P \in \mathcal{P}\} \le \inf\{U(f,P) \mid P \in \mathcal{P}\}$$

Then f is Riemann integrable on $[a,b]$ when equality occurs, i.e.,

$$\sup\{L(f,P) \mid P \in \mathcal{P}\} = \inf\{U(f,P) \mid P \in \mathcal{P}\}$$

This number is Riemann integral of f on $[a,b]$, denoted $\int_a^b f(x)\,dx$ or $\int_a^b f\,dx$ or simply $\int_a^b f$. Here are two problems that require knowing these definitions.

29 (January 2016) Give an example of a bounded function $g : [0,1] \to \mathbb{R}$ that is not Riemann integrable on $[0,1]$. Justify your answer based on the definition of Riemann integration.

Proof. Let us first review the definition of Riemann integrability on $[0,1]$. Let f be a bounded function defined on the interval $[0,1]$, and let $\mathcal{P}$ be the collection of all possible partitions of $[0,1]$. Then f is Riemann-integrable if and only if

$$\sup\{L(f,P) \mid P \in \mathcal{P}\} = \inf\{U(f,P) \mid P \in \mathcal{P}\}, \tag{1}$$

where for a partition $P = \{0 = x_0, \ldots, x_n = 1\}$ of $[0,1]$,

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}), \quad m_k = \inf\{f(x) \mid x_{k-1} \le x \le x_k\}$$
$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}), \quad M_k = \sup\{f(x) \mid x_{k-1} \le x \le x_k\}.$$

To violate (1), we use Dirichlet's function:

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Let P be a partition of $[0,1]$. Because both the rationals and the irrationals are dense in $\mathbb{R}$, we have $m_k = 0$ and $M_k = 1$ for all k, which easily implies $L(g,P) = 1$ and $U(g,P) = 0$. Since this is true for all partitions P, we get

$$\sup\{L(g,P) \mid P \in \mathcal{P}\} = 0 < 1 = \inf\{U(f,P) \mid P \in \mathcal{P}\},$$

so g is not Riemann integrable on $[0,1]$. QED

Notice how this solution gives a careful review of the definition of Riemann integrability at the beginning of the proof. Here is another way to handle the definition.

30 (March 2015) Let $f : [a,b] \to \mathbb{R}$ be the constant function given by

$$f(x) = c$$

for all x in $[0,1]$. Use the definition of Riemann integral to show that

$$\int_a^b f\,dx = c(b-a).$$

Proof. Let $P = \{a = x_0, \ldots, x_n = b\}$ be a partition of $[a,b]$. Since f is the constant function $f(x) = c$, $\inf\{f(x) \mid x_{k-1} \le x \le x_k\} = c$ for all k. Then the definition of lower sum implies

$$\begin{aligned} L(f,P) = \sum_{k=1}^{n} c(x_k - x_{k-1}) &= c\sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= c\big((x_1 - x_0) + (x_2 - x_1) + \cdots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})\big) \\ &= c(x_n - x_0) = c(b-a). \end{aligned}$$

In a similar way, $\sup\{f(x) \mid x_{k-1} \le x \le x_k\} = c$ implies $U(f,P) = c(b-a)$. Since this is true for every partition P, we obtain

$$\sup\{L(g,P) \mid P \in \mathcal{P}\} = c(b-a) = \inf\{U(f,P) \mid P \in \mathcal{P}\},$$

which proves that f is Riemann integrable on $[a,b]$ with integral $\int_a^b f = c(b-a)$. QED

Properties of the Riemann Integral. Know:

- Sums and constant multiples of Riemann integrable functions are Riemann integrable, with the formulas
$$\int_a^b f + g\,dx = \int_a^b f\,dx + \int_a^b g\,dx, \qquad \int_a^b cf\,dx = c\int_a^b f\,dx.$$
- If f, g are Riemann integrable on $[a,b]$ and $f(x) \ge g(x)$ for all $x \in [a,b]$, then $\int_a^b f\,dx \ge \int_a^b g\,dx$.

Integrability of a Continuous Function over $[a,b]$***.*** Know that if f is continuous on $[a,b]$, then it is Riemann integrable on $[a,b]$, so the integral $\int_a^b f\,dx$ exists. Uniform continuity plays a key role in the proof.

Integration of Sequences and Series. Know:

- If f_n is a Riemann integrable function on $[a,b]$ for all $n \ge 1$ and the sequence (f_n) converges uniformly to f on $[a,b]$, then f is Riemann integrable on $[a,b]$ and
$$\lim_{n\to\infty}\int_a^b f_n\,dx = \int_a^b \lim_{n\to\infty} f_n\,dx = \int_a^b f\,dx.$$

- If f_n is a Riemann integrable function on $[a,b]$ for all $n \geq 1$ and the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a,b]$, then f is Riemann integrable on $[a,b]$ and

$$\sum_{n=1}^{\infty} \int_a^b f_n \, dx = \int_a^b \sum_{n=1}^{\infty} f_n \, dx = \int_a^b f \, dx.$$

In the sequence case, uniform convergence allows you to move the limit inside of the integral, while in the series case, uniform convergence allows you to move the infinite sum inside of the integral. A lot of analysis was developed in order to determine when such manipulations were legal.

31 (March 2005) For $n \geq 1$, define the function g_n on $[0,1]$ by

$$g_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{2n} \\ n - n^2 x & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1. \end{cases}$$

(a) Draw the graph of $g_n(x)$.
(b) Compute $\lim_{n\to\infty} g_n(x)$ for all $x \in [0,1]$.
(c) Compute $\lim_{n\to\infty} \int_0^1 g_n(x)\, dx$.
(d) Show that $\{g_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0,1]$. Justify your answer using either definitions or standard theorems.

Hints. For part (a), you need to recognize the equation of a straight line, and for part (b), you need to know the area of a triangle. As for part (d), it is possible to show directly from the definition that $\{g_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0,1]$. However, it is much easier to use the fact that uniform convergence would imply

$$\lim_{n\to\infty} \int_0^1 g_n(x)\, dx = \int_0^1 \lim_{n\to\infty} g_n(x)\, dx.$$

Integration of Power Series. If a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$ and $[\alpha, \beta] \subseteq (a-R, a+R)$, then $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is continuous on $[\alpha,\beta]$ and

$$\int_\alpha^\beta f\, dx = \int_\alpha^\beta \sum_{n=0}^{\infty} a_n(x-a)^n \, dx = \sum_{n=0}^{\infty} a_n \int_\alpha^\beta (x-a)^n \, dx.$$

The integrals on the right are easy to evaluate by the Fundamental Theorem of Calculus.

11 Preparing for the Analysis Exam

Now that you have finished reading the content part of the Study Guide, what should you do next to prepare for the analysis exam? The key thing to keep in mind is that

You need an active knowledge of analysis.

Here are a some suggestions to help you achieve this.

Read the Study Guide Actively. There are many places where the Study Guide asks you to provide a proof or complete a proof. Do so. In other places, you are asked to understand the similarities and differences between a group of problems. Do so.

Read Your Notes and Your Analysis Book. In many places in the Study Guide, we say "Know the basic facts about ...", without stating the facts precisely. This is deliberate, since we want you to refer to your notes and your analysis book when studying for the exam.

If you do not have a copy of your analysis book, note that *Understanding Analysis* by Stephen Abbott is used frequently in Math 355. You can download a pdf copy of this book from the Amherst College Library.

Not everything covered in your analysis course is part of the analysis exam. For example, countable and uncountable sets are a lovely topic that will not be on the exam. This Study Guide and the *Syllabus for Analysis* (*Math 355*) list the topics that you need to know.

Know Basic Results and Definitions. Keep in mind that knowing the precise statements of definitions and basic theorems is essential. The adjective "precise" is important here. For example, if a problem asks you to state the Mean Value Theorem, then just writing

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

will not get full credit. You need to state the whole theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Study Old Exams and Solutions. The Department website has a collection of old exams in analysis, many with solutions. It is very important to do practice problems. This is one of the key ways to acquire an active knowledge of analysis. There are two dangers to be aware of when using old exams and solutions:

- Thinking that the exams tell you what to study. Every topic on the *Syllabus* and in this Study Guide is fair game for an exam question.
- Reading the solutions. This is passive. To get an active knowledge of the material, do problems from the old exams yourself, and *then* check the solutions. The more you can do this, the better.

Work Together, Ask Questions, and Get Help. Studying with your fellow math majors can help. You can learn a lot from each other. Faculty are delighted to help. Don't hesitate to ask us questions and show us your solutions so we can give you feedback. The QCenter has excellent people who have helped many students in the past prepare for the Comprehensive Exam.

Start Now. Properly preparing for the Analysis Exam will take longer than you think. Start now.