AMHERST COLLEGE

Department of Mathematics and Statistics

COMPREHENSIVE EXAMINATION

Multivariable Calculus and Linear Algebra

2:00 pm Friday, January 30, 2015

206 Seeley Mudd

There are 9 problems (10 points each, totaling 90 points) on this portion of the examination. Record your answers in the blue book provided. **Show all of your work.**

1. (10 points) Find all critical points of the function $f(x, y) = x^3 + y^3 - 3xy$, and classify each as a local minimum, local maximum, or saddle point.

2. Let C be the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$, oriented counterclockwise, and let $\mathbf{F}(x, y) = \langle xy, x^2 \rangle$.

   (a) (4 points) According to Green's theorem, the line integral

   $$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C xy\, dx + x^2\, dy$$

   is equal to a certain double integral. Set up this double integral.

   (b) (6 points) Verify Green's theorem in this case by evaluating both the line integral and the double integral in part (a).

3. Let

   $$f(x, y) = \begin{cases} \dfrac{x^4 + y^3 + xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

   (a) (4 points) Is f continuous at $(0, 0)$? Justify your answer.

   (b) (6 points) Find $f_x(0, 0)$ and $f_y(0, 0)$.

4. (10 points) Find the directional derivative of the function $f(x, y, z) = x\sqrt{yz + 1}$ at the point $(2, 1, 3)$ in the direction of the vector $\langle 2, -1, 2 \rangle$.

5. (10 points) Find the volume of the region that is inside both the sphere $x^2+y^2+z^2 = 25$ and the cylinder $x^2 + y^2 = 9$.

6. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$, where

   $$A = \begin{bmatrix} 1 & -2 & 2 & 7 \\ 2 & -4 & 2 & 8 \\ 1 & -2 & -1 & -2 \end{bmatrix}.$$

   (a) (7 points) Find a basis for the kernel, or nullspace, of T.

   (b) (3 points) What is the dimension of the range of T?

7. (10 points) Suppose that V is a vector space and u and v are vectors in V. Show that $\text{Span}(\{3u + v, u - v\}) = \text{Span}(\{u, v\})$.

8. Let

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

   (a) (5 points) Find all eigenvalues of A.

   (b) (5 points) Find, if possible, an invertible matrix P such that $P^{-1}AP$ is diagonal, or show that no such matrix exists.

9. (10 points) Let V by the vector space of polynomials of degree at most 2, and let $\mathcal{B} = \{1, x+1, x^2+x+1\}$, which is a basis for V. Suppose that $T : V \to V$ is a linear transformation, and the matrix of T relative to $\mathcal{B}$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 5 & 1 \end{bmatrix}.$$

   Find $T(3x^2 + x + 2)$.

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Comprehensive Examination: Algebra

Friday, January 30, 2015

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(25 points).** Let G and H be groups, let $B \subseteq H$ be a subgroup, and let $\phi : G \to H$ be a homomorphism. Define

$$A = \{g \in G \mid \phi(g) \in B\}.$$

(A is called the *inverse image* of B under ϕ.) Prove that A is a subgroup of G.

[Note: this is a standard theorem in Math 350. You are being asked to prove this theorem, so obviously you cannot simply quote that theorem.]

2. **(25 points).** Let G be an abelian group, and define

$$T = \{g \in G \mid g \text{ has finite order}\}.$$

It is a fact, which you may assume, that T is a normal subgroup of G.

Prove that the only element of the quotient group G/T that has finite order is the identity element.

3. **(25 points).** Recall that S_n denotes the group of permutations of the set $\{1, 2, \ldots, n\}$.

   (a) Let $\sigma = (1, 4, 3)(3, 5)(2, 7, 5)(1, 6, 2, 4, 7) \in S_7$. Write σ as a product of disjoint cycles.

   (b) Determine whether σ is an even or an odd permutation.

   (c) Give an example of an **even** permutation $\tau \in S_7$ of order 4. Don't forget to justify your answer.

4. **(25 points).** Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R. If you use any other technical terms like "closed," "subring," "subgroup," etc., you must fully define those terms as well.

   (b) Assume R is commutative, and let $I, J \subseteq R$ be ideals of R. Define

$$IJ = \{x_1y_1 + \cdots + x_ny_n \mid n \geq 1 \text{ and } x_i \in I,\ y_i \in J\}.$$

   That is, IJ is the set of all finite sums of products of an element of I times an element of J. Prove that IJ is an ideal of R.

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Comprehensive Examination: Analysis
January 30, 2015

Answer all problems in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. (a) (2 points) Explain what it means to say that a sequence (x_n) of real numbers is *monotone*.

   (b) (2 points) Give a precise statement of the Monotone Convergence Theorem.

   (c) (2 points) Give an example of a monotone sequence of real numbers that does not converge.

2. (a) (4 points) State the ϵ/δ definition of what it means for the function f to be continuous at c.

   (b) (6 points) Suppose that f and g are functions that are continuous at c. Prove that the function $(f+g)$, given by

   $$(f+g)(x) = f(x) + g(x),$$

   is also continuous at c.

3. (a) (6 points) Show that the sequence of functions (x^n) converges pointwise, as $n \to \infty$, for x in the interval $[0,1]$.

   (b) (4 points) Explain precisely what it means to say that a sequence (f_n) of functions $f_n : [0,1] \to \mathbb{R}$ converges uniformly to a function f.

   (c) (4 points) Does the sequence (x^n) converge uniformly on $[0,1]$? Explain your answer.

4. (a) (3 points) State the Mean Value Theorem as it applies to a function f. (Be sure to include all the necessary hypotheses on the function f.)

   (b) (7 points) Suppose a function f has the following properties:

   - the domain of f is $\mathbb{R}$;
   - f is differentiable at every real number;
   - $f'(x) = 0$ for every real number x.

   Prove that there is a constant K such that $f(x) = K$ for every real number x. (Hint: show that $f(x) = f(0)$ for all x.)