1. [25 points] Find two points on the ellipsoid $x^2 + 2y^2 + 4z^2 = 10$ where the tangent plane is perpendicular to the vector $\langle 1,\, 1,\, -2\rangle$.

2. [25 points] Let $f(x,y) = \begin{cases} \dfrac{4x^2+3xy+2y^2}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0). \end{cases}$

   (a) [15 points] Compute $f_x(0,0)$ and $f_y(0,0)$.

   (b) [10 points] Prove that f is **not** continuous at $(0,0)$.

3. [25 points] Find the points at which the absolute maximum and minimum of the function $f(x,y) = xy - 1$ on the disk $x^2 + y^2 \leq 2$ occur. State all points where the extrema occur as well as the maximum and minimum values.

4. [25 points] Consider the paraboloid $z = x^2 + y^2$, which is intersected by the plane $z = 4$.

   (a) [15 points] Find the volume of the region that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$.

   (b) [10 points] Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that is below the plane $z = 4$.

5. [25 points] Let V be the vector space of polynomials of degree at most 2 and let $T : V \to V$ be the mapping given by

$$T(ax^2 + bx + c) = (b - a)x^2 + (c - b)x + (a - c).$$

   (a) [10 points] Prove that T is a linear transformation.

   (b) [10 points] Calculate the dimension of the null space, or kernel, of T.

   (c) [5 points] Calculate the dimension of the image space, or range, of T.

6. [30 points]

   (a) [15 points] Find a basis for the subspace of $\mathbb{R}^3$ given by the span of the following set of vectors:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

   (b) [15 points] Give an example of a vector in $\mathbb{R}^3$ that is not in the subspace in part (a). Justify your answer.

7. [20 points] Suppose that V is a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Prove that the set

$$\{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1\}$$

   is also a basis for V.

8. [25 points]

   (a) [10 points] Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

   (b) [15 points] Show that the matrix A is not diagonalizable.

1. [25 points] Let G be a group, let $H \subseteq G$ be a subgroup, and define the set K to be

$$K = \{x \in G \mid Hx = xH\}.$$

   (a) [17 points] Prove that K is a subgroup of G.

   (b) [8 points] Prove that $H \subseteq K$.

2. [25 points] Let G be a group, and let $N \subseteq G$ be a normal subgroup. Prove that

$$\text{the quotient group } G/N \text{ is abelian}$$

   if and only if

$$\text{for every } x, y \in G, \text{ we have } xyx^{-1}y^{-1} \in N.$$

3. [25 points] Consider the group S_{100} of permutations of the set $\{1, 2, 3, \ldots, 100\}$. Let $\sigma \in S_{100}$ be the permutation

$$\sigma = (1\ 3\ 2)(3\ 6)(1\ 4\ 3\ 5)(2\ 3\ 6\ 5\ 4).$$

   (a) [8 points] Write σ as a product of **disjoint** cycles.

   (b) [7 points] Compute the **order** of σ.

   (c) [10 points] For each integer $n = 7, 8, \ldots, 100$, let τ_n be the 4-cycle $\tau_n = (1\ n\ 2\ 4)$. For each such n, decide whether the product $\sigma\tau_n$ is an **even** or **odd** permutation.

4. [25 points] Let R be a ring.

   (a) [8 points] Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
   Note: If you use other terms like "closed," "subring," "subgroup," etc., you must fully define those terms as well.

   (b) [17 points] Let S be another ring, and let $\phi : R \to S$ be a ring homomorphism. Let $I \subseteq R$ be an ideal of R, and define

$$J = \{x \in I \mid \phi(x) = 0_S\},$$

   where 0_S denotes the zero element of S. Give a complete proof that J is an ideal of R.

1. [25 points]
   (a) [5 points] What does it mean to say that a sequence (a_n) of real numbers *converges* to the real number a?
   (b) [5 points] What does it mean to say that a sequence (a_n) of real numbers is a *Cauchy sequence*?
   (c) [15 points] Let (a_n) be a convergent sequence of real numbers. Prove that (a_n) is a Cauchy sequence.

2. [25 points]
   (a) [5 points] State the ϵ/δ definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous* at c.
   (b) [20 points] Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that g is continuous at c and f is continuous at $g(c)$. Prove that the composite function $f \circ g$, defined by

   $$(f \circ g)(x) = f(g(x)),$$

   is continuous at c. (If your proof does not use the definition given in part (a), be sure to state clearly the property of continuity you are using and explain how it is being used.)

3. [30 points]
   (a) [5 points] What does it mean to say that a power series $\sum_{n=0}^{\infty} c_n x^n$ *converges uniformly* on an interval $[a, b]$?
   (b) [15 points] Prove that the power series

   $$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

   converges uniformly on the interval $[-r, r]$ for any $0 < r < 1$. (You may use the Weierstrass M-Test if you state the test precisely and explain clearly how it applies.)
   (c) [10 points] Does the power series in part (b) converge uniformly on $[-1, 1]$? Justify your answer.

4. [20 points] Do EITHER part (a) OR part (b), NOT BOTH.
   (a) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ that is not Riemann-integrable on the interval $[0, 1]$. Justify your answer based on the definition of Riemann integration.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f has a local maximum at c. Prove that $f'(c) = 0$.