Comps Study Guide for Multivariable Calculus

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This Study Guide was written to help you prepare for the multivariable calculus portion of the Comprehensive and Honors Qualifying Examination in Mathematics. It is based on the *Syllabus for the Comprehensive Examination in Multivariable Calculus* (*Math 211*) available on the Department website.

Each topic from the syllabus is accompanied by a brief discussion and examples from old exams. When reading this guide, you should focus on three things:

- *Understand the ideas.* If you study problems and solutions without understanding the underlying ideas, you will not be prepared for the exam.
- *Understand the strategy of each problem.* Most solutions in this guide are short—the hardest part is often knowing where to start. Focus on this rather than falling into the trap of memorizing solutions.
- *Understand the value of scratchwork.* Brainstorm possible solution methods and draw pictures when relevant to help you identity a good approach to the problem.

The final section of the guide has some further suggestions for how to prepare for the exam.

1 Elementary Vector Analysis

Most of multivariable calculus takes place in $\mathbb{R}^2$ and $\mathbb{R}^3$. You should be familiar with the Cartesian coordinates $(x, y) \in \mathbb{R}^2$ and $(x, y, z) \in \mathbb{R}^3$.

Vectors. A vector $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is often represented by a directed line segment. In term of coordinates, we write $\mathbf{v} = \langle a_1, a_2\rangle$ in $\mathbb{R}^2$ and $\mathbf{v} = \langle a_1, a_2, a_3\rangle$ in $\mathbb{R}^3$. Know:

- Addition and scalar multiplication of vectors.
- The standard basis vectors $\mathbf{i} = \langle 1, 0\rangle$, $\mathbf{j} = \langle 0, 1\rangle$ in $\mathbb{R}^2$ and $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$ in $\mathbb{R}^3$.
- A vector $\mathbf{v}$ has length $|\mathbf{v}|$, sometimes denoted $||\mathbf{v}||$.
- Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if each is a constant multiple of the other.
- A point P in $\mathbb{R}^2$ or $\mathbb{R}^3$ gives a vector from the origin to P, called the position vector of P. This allows us to regards points as vectors and vice versa.

Also know the formula for $|\mathbf{v}|$ and how it relates to the distance formula for the distance between two points in $\mathbb{R}^2$ or $\mathbb{R}^3$. See $\boxed{\mathbf{12}}$ for a problem that uses the distance formula and $\boxed{\mathbf{20}}$ for a problem that uses the length of a vector. Note that vectors are sometimes written $\vec{v}$ instead of $\mathbf{v}$.

Dot Product. In $\mathbb{R}^2$, the dot product of $\mathbf{u} = \langle a_1, a_2\rangle$ and $\mathbf{v} = \langle b_1, b_2\rangle$ is $\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2$, and similarly, in $\mathbb{R}^3$, the dot product of $\mathbf{u} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{v} = \langle b_1, b_2, b_3\rangle$ is $\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2 + a_3b_3$. Know:

- Linearity properties of dot product.

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (orthogonal).
- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.

Dot product is sometimes called the scalar product. See $\boxed{\mathbf{3}}$ and $\boxed{\mathbf{4}}$ for problems that use dot product.

Cross Product. Given $\mathbf{u} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{v} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$, their cross product is

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Know:

- Linearity properties of cross product.
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u} \times \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Cross product is sometimes called the vector product. See $\boxed{\mathbf{6}}$ for a problem that uses cross product.

Lines and Planes. Know:

- In $\mathbb{R}^2$ or $\mathbb{R}^3$, a point $\mathbf{r}_0$ and a nonzero vector $\mathbf{v}$ determine the line parametrized by

  $$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}.$$

  The vector $\mathbf{v}$ is called a direction vector of the line. Be sure you know how to write out the parametric equations of a line for the coordinates $(x, y) \in \mathbb{R}^2$ or $(x, y, z) \in \mathbb{R}^3$.

- A plane in $\mathbb{R}^3$ is defined by an equation of the form $ax + by + cz = d$ where $\langle a, b, c \rangle \neq \langle 0, 0, 0 \rangle$. A more geometric way to write the equation uses a nonzero vector $\mathbf{n}$ perpendicular to the plane and point (x_0, y_0, z_0) in the plane. Then:

  $$\begin{aligned} (x, y, z) \text{ is in the plane} &\iff \mathbf{n} \text{ is perpendicular to the vector from } (x, y, z) \text{ to } (x_0, y_0, z_0) \\ &\iff \mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0. \end{aligned}$$

  The vector $\mathbf{n}$ is called a normal vector to the plane. For a plane defined by $ax + by + cz = d$, a normal vector is given by $\mathbf{n} = \langle a, b, c \rangle$.

Here is a problem that uses lines and planes.

$\boxed{1}$ (January 2017) Find an equation of the form $ax + by + cz = d$ for the plane passing through the point $(-2, -1, 4)$ that is perpendicular to the line with parametric equations $x = 2t,\ y = 3t - 1,\ z = 5 - t$.

Solution. Since the line is perpendicular to the plane, its direction vector is a normal vector to the plane. Be sure you can draw picture of this. Writing the line as

$$\mathbf{r}(t) = (2t, 3t-1, 5-t) = (0,-1,5) + t\,\langle 2,3,-1\rangle,$$

we see that $\langle 2,3,-1\rangle$ is a normal vector to the plane. Hence the equation of the plane can be written

$$2x + 3y - z = d$$

for some $d \in \mathbb{R}$. Rereading the problems shows that there is further information, namely that the plane passes through $(-2,-1,4)$. Thus this point satisfies the above equation, i.e.,

$$2(-2) + 3(-1) - (4) = d.$$

This implies $d = -11$ and the equation is $2x + 3y - z = -11$.

Comment. Drawing a picture of a line perpendicular to a plane can help clarify the geometry of the problem and lead you to the right solution. A good strategy is to draw pictures first, rather than immediately jumping into formulas and equations.

Tangent Vector to a Parametrized Curve. Given a curve parametrization $\mathbf{r}(t) = (x(t), y(t))$ in the plane, the tangent vector to the curve at the point $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle.$$

The situation is similar on $\mathbb{R}^3$. Here is a typical problem.

$\boxed{\mathbf{2}}$ (March 2017) Find parametric equations for the line that is tangent to the curve given by

$$x = t^2 - 2t - 1, \quad y = t^4 - 4t^2 + 2$$

at the point $(-2,-1)$.

Solution. The tangent vector to $\mathbf{r}(t) = (t^2 - 2t - 1, t^4 - 4t^2 + 2)$ is $\mathbf{r}'(t) = \langle 2t-2, 4t^3 - 8t\rangle$. Since we want the tangent line at $(-2,-1)$, we need to find $t \in \mathbb{R}$ such that $\mathbf{r}(t) = (-2,-1)$. Be sure you understand this. To solve $(t^2 - 2t - 1, t^4 - 4t^2 + 2) = (-2,-1)$, we begin with the x-coordinate:

$$t^2 - 2t - 1 = -2 \;\Rightarrow\; t^2 - 2t + 1 = 0 \;\Rightarrow\; (t-1)^2 = 0 \;\Rightarrow\; t = 1.$$

Then one computes that $\mathbf{r}(1) = (1^2 - 2\cdot 1 - 1, 1^4 - 4\cdot 1^2 + 2) = (1-2-1, 1-4+2) = (-2,-1)$ and $\mathbf{r}'(1) = \langle 2\cdot 1 - 2, 4\cdot 1^3 - 8\cdot 1\rangle = \langle 2-2, 4-8\rangle = \langle 0,-4\rangle$. Since the tangent line goes through $\mathbf{r}(1)$ with direction vector $\mathbf{r}'(1)$, the tangent line is parametrized by

$$\mathbf{r}(1) + t\,\mathbf{r}'(1) = (-2,-1) + t\,\langle 0,-4\rangle = (-2, -4t-1), \text{ i.e., } x = -2,\ y = -4t - 1.$$

2 Functions of Several Variables

Partial Derivatives. Know:

- The definition of partial derivative of a function $f(x,y)$ or $f(x,y,z)$.

- The standard notation for the partial derivatives: $\dfrac{\partial f}{\partial x} = f_x(x,y)$, $\dfrac{\partial f}{\partial y} = f_y(x,y)$, $\dfrac{\partial^2 f}{\partial^2 x} = f_{xx}(x,y)$, $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}(x,y)$, $\dfrac{\partial^2 f}{\partial^2 y} = f_{yy}(x,y)$ for $f(x,y)$, and similarly for $f(x,y,z)$.
- The rate of change interpretation of a partial derivative.
- How to compute partial derivatives using the standard rules of differentiation.

Directional Derivatives. Know:

- The definition of a unit vector $\mathbf{u}$ and how to rescale a nonzero vector to make it a unit vector.
- The definition of the directional derivative $D_{\mathbf{u}} f(a,b)$ of $f(x,y)$ in the direction of the unit vector $\mathbf{u}$ at the point (a,b), and similarly for $f(x,y,z)$.
- The rate of change interpretation of a directional derivative.

Also know the theorem (stated below) that computes the directional derivative using the gradient when the function is differentiable.

The Gradient. The gradient of $f(x,y)$ at (a,b) is the vector

$$\nabla f(a,b) = \frac{\partial f}{\partial x}(a,b)\mathbf{i} + \frac{\partial f}{\partial y}(a,b)\mathbf{j} = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle,$$

and similarly for $f(x,y,z)$. Know:

- $\nabla f(a,b)$ is perpendicular to the level curve $f(x,y) = f(a,b)$ at the point (a,b). Similarly, $\nabla f(a,b,c)$ is perpendicular to the level surface $f(x,y,z) = f(a,b,c)$ at (a,b,c).
- If $f(a,b)$ is differentiable at (a,b) and $\mathbf{u}$ is a unit vector, then

  $$D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u},$$

  and similarly for $f(x,y,z)$.
- When $\nabla f(a,b) \neq \mathbf{0}$, the unit vector $\nabla f(a,b)/|\nabla f(a,b)|$ gives the direction in which $f(x,y)$ is increasing most rapidly. Furthermore, the maximum rate of increase is $|\nabla f(a,b)|$. Similar results hold for $f(x,y,z)$.

Here are two problems that feature the gradient. See also $\boxed{6}$ and $\boxed{20}$.

$\boxed{3}$ (January 2015) Find the directional derivative of the function $f(x,y,z) = x\sqrt{yz+1}$ at the point $(2,1,3)$ in the direction of the vector $\langle 2,-1,2 \rangle$.

Solution. The unit vector in the direction of $\langle 2,-1,2 \rangle$ is

$$\mathbf{u} = \frac{\langle 2,-1,2 \rangle}{|\langle 2,-1,2 \rangle|} = \frac{\langle 2,-1,2 \rangle}{\sqrt{2^2+(-1)^2+2^2}} = \frac{\langle 2,-1,2 \rangle}{3} = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle.$$

The directional derivative of $f = x\sqrt{yz+1}$ in the direction of $\mathbf{u}$ is therefore

$$\begin{aligned}
D_{\mathbf{u}} f(x,y,z) &= \nabla f(x,y,z) \cdot \mathbf{u} \\
&= \left\langle \frac{\partial}{\partial x} x\sqrt{yz+1}, \frac{\partial}{\partial y} x\sqrt{yz+1}, \frac{\partial}{\partial z} x\sqrt{yz+1} \right\rangle \cdot \mathbf{u} \\
&= \left\langle \sqrt{yz+1}, \frac{xz}{2\sqrt{yz+1}}, \frac{xy}{2\sqrt{yz+1}} \right\rangle \cdot \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle \\
&= \frac{2}{3}\sqrt{yz+1} - \frac{xz}{6\sqrt{yz+1}} + \frac{xy}{3\sqrt{yz+1}}.
\end{aligned}$$

Then setting $(x,y,z) = (2,1,3)$ gives

$$D_{\mathbf{u}} f(2,1,3) = \frac{2}{3}\sqrt{1\cdot 3+1} - \frac{2\cdot 3}{6\sqrt{1\cdot 3+1}} + \frac{2\cdot 1}{3\sqrt{1\cdot 3+1}} = \frac{4}{3} - \frac{1}{2} + \frac{1}{3} = \frac{7}{6}.$$

4 (January 2011) Let $f(x,y)$ be differentiable on $\mathbb{R}^2$. Suppose that $f_x(0,0) = 2$ and that the directional derivative of f at $(0,0)$ in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}(1,1)$ is $5/\sqrt{2}$. Determine the value of $f_y(0,0)$.

Solution. By the formula for the directional derivative,

$$D_{\mathbf{u}} f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = f_x(0,0)\,\frac{1}{\sqrt{2}} + f_y(0,0)\,\frac{1}{\sqrt{2}}$$

since $\mathbf{u} = \frac{1}{\sqrt{2}}(1,1)$. From the given conditions we know $D_{\mathbf{u}} f(0,0) = \frac{5}{\sqrt{2}}$ and $f_x(0,0) = 2$. Substituting these numbers into the above equation yields

$$\frac{5}{\sqrt{2}} = 2\,\frac{1}{\sqrt{2}} + f_y(0,0)\,\frac{1}{\sqrt{2}},$$

from which we conclude that $f_y(0,0) = 3$.

Here is a problem you should do yourself.

5 (March 2009) The temperature at the point (x,y,z) is

$$T(x,y,z) = \frac{1}{\pi}\sin(\pi xy) + \ln(z^2+1) + 60.$$

(a) Find a vector pointing in the direction in which the temperature increases most rapidly at the point $(2,-1,1)$. ANSWER: $\langle -1, 2, 1 \rangle$

(b) Let $\vec{v} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. (Notice that $\vec{v}$ is **not** a unit vector.) What is the rate of change of the temperature at the point $(2,-1,1)$ in the direction of $\vec{v}$? ANSWER: $\frac{7}{3}$

The Tangent Plane to a Surface. Tangent planes arise in two situations:

- If $f(x,y)$ is differentiable at (x_0,y_0), then the tangent plane to the graph $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$ is defined by

(1) $$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

- If $F(x,y,z)$ is differentiable at (x_0,y_0,z_0), then (x_0,y_0,z_0) lies on the level surface $F(x,y,z) = F(x_0,y_0,z_0)$, and the equation of the tangent plane to the surface at this point is defined by

$$\nabla F(x_0,y_0,z_0)\cdot(x-x_0,y-y_0,z-z_0)=0, \tag{2}$$

provided that the gradient $\nabla F(x_0,y_0,z_0)$ is nonzero. Written out, this is the equation

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

The two situations are related since the graph $z=f(x,y)$ is the level surface $F(x,y,z)=f(x,y)-z=0$. Since $\nabla F = f_x\,\mathbf{i}+f_y\,\mathbf{j}-\mathbf{k}$, equation (2) reduces to equation (1) in this case.

Here are two problems that involve tangent planes.

$\boxed{6}$ (March 2007) Let $F(x,y,z)=xy^2z^3$.
(a) Find the equation of the tangent plane to the level surface $F(x,y,z)=1$ at the point $(1,1,1)$.
(b) Compute $\nabla F(1,1,1)\times\vec{v}$, where $\vec{v}=(2,-1,3)$.

Solution. (a) Since we need the gradient for part (b) and the gradient is normal to the tangent plane, it make sense to start with the gradient:

$$\nabla F = \left\langle \frac{\partial}{\partial x}(xy^2z^3), \frac{\partial}{\partial y}(xy^2z^3), \frac{\partial}{\partial z}(xy^2z^3)\right\rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle.$$

Then $\nabla F(1,1,1)=\langle 1,2,3\rangle$. Since the given point is $(1,1,1)$, the equation of the tangent plane is

$$1\cdot(x-1)+2\cdot(y-1)+3\cdot(z-1)=0 \;\Rightarrow\; x+2y+3z=6.$$

(b) Be sure you can do this straightforward computation. Answer: $9\,\mathbf{i}+3\,\mathbf{j}-5\,\mathbf{k}$

$\boxed{7}$ (January 2014) Suppose the plane $z=2x-y-1$ is tangent to the graph of $z=f(x,y)$ at $P=(5,3)$. Determine $f(5,3)$, $\dfrac{\partial f}{\partial x}(5,3)$ and $\dfrac{\partial f}{\partial y}(5,3)$.

Solution. In general, the tangent plane to the surface $z=f(x,y)$ at the point $(x_0,y_0,f(x_0,y_0))$ is defined by the equation

$$z-f(x_0,y_0)=\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0,)(y-y_0),$$

which for $(x_0,y_0)=(5,3)$ reduces to

$$\begin{aligned} z &= \frac{\partial f}{\partial x}(5,3)(x-5)+\frac{\partial f}{\partial y}(5,3)(y-3)+f(5,3)\\ &= \frac{\partial f}{\partial x}(5,3)\,x+\frac{\partial f}{\partial y}(5,3)\,y+\big(f(5,3)-5\frac{\partial f}{\partial x}(5,3)-3\frac{\partial f}{\partial y}(5,3)\big). \end{aligned}$$

However, the problem tells us that the tangent plane at $(5,3,f(5,3))$ is

$$z=2x-y-1.$$

Comparing coefficients of x and y, we obtain

$$\frac{\partial f}{\partial x}(5,3)=2 \qquad \frac{\partial f}{\partial y}(5,3)=-1.$$

Then comparing constant terms gives

$$-1 = f(5,3) - 5\frac{\partial f}{\partial x}(5,3) - 3\frac{\partial f}{\partial y}(5,3) = f(5,3) - 5 \cdot 2 - 3 \cdot (-1) = f(5,3) - 7,$$

so that $f(5,3) = 6$. Or, more simply, substitute $x = 5$ and $y = 3$ into the given tangent plane equation to obtain $z = 2 \cdot 5 - 3 - 1 = 6$.

3 Maxima and Minima of Functions of Several Variables

Finding Critical Points. In two dimensions, (a,b) is a critical point of $f(x,y)$ provided

$$f_x(a,b) = f_y(a,b) = 0.$$

Also know the definition in three dimensions. Here is a problem involving critical points.

8 (January 2011) Let $f(x,y) = 4xy - x^4 - y^4$. Find the critical points of $f(x,y)$.

Solution. We need to solve the equations

$$\frac{\partial}{\partial x}\left(4xy - x^4 - y^4\right) = 4y - 4x^3 = 0 \;\Rightarrow\; y = x^3$$
$$\frac{\partial}{\partial y}\left(4xy - x^4 - y^4\right) = 4x - 4y^3 = 0 \;\Rightarrow\; x = y^3.$$

Substituting the first equation into the second gives

$$x = (x^3)^3 = x^9 \;\Rightarrow\; x - x^9 = 0 \;\Rightarrow\; x(1 - x^8) = 0.$$

Factoring further, we obtain

$$0 = x(1-x^8) = x(1-x^4)(1+x^4) = x(1-x^2)(1+x^2)(1+x^4) = x(1-x)(1+x)(1+x^2)(1+x^4).$$

The last two factors never vanish, so $x = 0, \pm 1$. Since $y = x^3$, we get three critical points

$$(0,0),\ (1,1),\ (-1,-1).$$

Comment. A common mistake is canceling a factor from an equation such as $x = x^9$. Here, canceling x would give $1 = x^8$, which loses the critical point $(0,0)$.

The Second Derivative Test for Local Maxima/Minima and Saddle Points. Know the definitions of local maximum and local minimum, and the fact that local maxima and minima occur at critical points when the function is differentiable. Also know the definition of saddle point.

For a suitably nice function $f(x,y)$, the second derivative test goes as follows. Let (a,b) be a critical point of f, and define

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Then:

If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b).
If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b).
If $D(a,b) < 0$, then f has a saddle point at (a,b).

The second derivative test is inconclusive in all other cases. Here is a typical problem.

$\boxed{9}$ (March 2011) Let $f(x,y) = xy^2 - 2x^2 - y^2$. Find all critical points of f, and classify them as local maxima, local minima, and saddle points.

Solution. To find the critical points, we need to solve the equations

$$f_x(x,y) = y^2 - 4x = 0, \quad f_y(x,y) = 2xy - 2y = 0,$$

which are equivalent to

$$x = \tfrac{1}{4}y^2, \quad y(x-1) = 0.$$

The second equation gives $y = 0$ or $x = 1$. We pursue each separately:

$$y = 0 \;\Rightarrow\; x = \tfrac{1}{4}0^2 = 0, \text{ giving } (x,y) = (0,0)$$
$$x = 1 \;\Rightarrow\; 1 = \tfrac{1}{4}y^2 \;\Rightarrow\; y^2 = 4 \;\Rightarrow\; y = \pm 2, \text{ giving } (x,y) = (1,\pm 2).$$

To classify the critical points $(0,0)$, $(1,\pm 2)$, we compute the second partials:

$$f_{xx}(x,y) = -4, \quad f_{xy}(x,y) = 2y, \quad f_{yy}(x,y) = 2x - 2,$$

so that

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \det\begin{pmatrix} -4 & 2y \\ 2y & 2x-2 \end{pmatrix} = -4(2x-2) - (2y)^2 = -8x + 8 - 4y^2.$$

Thus

$$D(0,0) = 8 > 0, f_{xx}(0,0) = -4 < 0 \;\Rightarrow\; \text{local maximum at } (0,0)$$
$$D(1,\pm 2) = -8 + 8 - 4(\pm 2)^2 = -16 < 0 \;\Rightarrow\; \text{saddle point at } (1,\pm 2).$$

Here is a similar problem you should do yourself for practice.

$\boxed{10}$ (March 2006) Locate the critical points of $f(x,y) = (x+y)^3 + 6(x^2 + y^2)$ and determine the type (local maximum, local minimum, saddle point) of each critical point.

Answer: local minimum at $(0,0)$, saddle point at $(-1,-1)$

The Method of Lagrange Multipliers. In a constrained optimization problem, you want to find the maximum or minimum of a function subject to a constraint. Such problems occur in two and three dimensions and use the method of Lagrange multipliers. We assume that the function and constraint are differentiable.

- To maximize or minimize $f(x,y)$ subject to the constraint $g(x,y) = 0$, solve

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 0,$$

or equivalently,

$$f_x(x,y) = \lambda g_x(x,y), \quad f_y(x,y) = \lambda g_y(x,y), \quad g(x,y) = 0.$$

- To maximize or minimize $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$, solve

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z), \quad g(x,y,z)=0,$$

or equivalently,

$$f_x(x,y,z) = \lambda g_x(x,y,z), \quad f_y(x,y,z) = \lambda g_y(x,y,z), \quad f_z(x,y,z) = \lambda g_z(x,y,z), \quad g(x,y,z)=0.$$

It is customary to call λ the Lagrange multiplier. Here are two problems that use Lagrange multipliers.

$\boxed{11}$ (January 2009) Let $f(x,y) = 2x+5y^2$. Find the maximum and minimum values of $f(x,y)$ on the curve $x^2+5y^4=9$.

Solution. By Lagrange multipliers, we need to solve the equations

$$2 = \lambda \cdot 2x, \quad 10y = \lambda \cdot 20y^3, \quad x^2+5y^4=9.$$

The first equation tells us that $\lambda x = 1$ (so $\lambda \neq 0$), while the second implies

$$y = 2\lambda y^3 \;\Rightarrow\; y - 2\lambda y^3 = 0 \;\Rightarrow\; y(1-2\lambda y^2) = 0 \;\Rightarrow\; y=0 \text{ or } 2\lambda y^2 = 1.$$

We pursue the two possibilities for y separately:

$y=0$: The constraint implies $x^2+9\cdot 0^2 = 9$, so that $x=\pm 3$. This gives the points $(\pm 3, 0)$.

$2\lambda y^2 = 1$: Here, there are two ways to proceed:

- (Systematic) Write x, y in terms of λ and substitute into $x^2+5y^4=9$. Since $x = \frac{1}{\lambda},\ y^2 = \frac{1}{2\lambda}$ (the constraint involves $y^4 = (y^2)^2$), we have

$$\left(\frac{1}{\lambda}\right)^2 + 5\left(\frac{1}{2\lambda}\right)^2 = 9 \;\Rightarrow\; \frac{1}{\lambda^2} + \frac{5}{4\lambda^2} = \frac{9}{4\lambda^2} = 9 \;\Rightarrow\; 4\lambda^2 = 1 \;\Rightarrow\; \lambda = \pm\frac{1}{2}.$$

  When $\lambda = \frac{1}{2}$, we get $x=2,\ y^2=1$, giving the points $(2,\pm 1)$. When $\lambda = -\frac{1}{2}$, we get $x=-2,\ y^2=-1$, which has no solutions over $\mathbb{R}$.

- (Clever) Since $\lambda = 1/x$, $2\lambda y^2 = 1$ implies $2y^2/x = 1$, so that $y^2 = x/2$. Substituting into the constraint gives

$$x^2 + 5\left(\frac{x}{2}\right)^2 = 9 \;\Rightarrow\; \frac{9x^2}{4} = 9 \;\Rightarrow\; x^2 = 4 \Rightarrow\; x = \pm 2.$$

  When $x=2$, we get $y^2 = 2/2 = 1$, giving the points $(2,\pm 1)$. When $x=-2$, we get $y^2 = (-2)/2 = -1$, which has no solutions over $\mathbb{R}$.

It follows that the maximum and minimum of $f = 2x+5y^2$ occur among $(\pm 3, 0), (2, \pm 1)$. Since

$$f(\pm 3, 0) = 2\cdot(\pm 3) + 5\cdot 0^2 = \pm 6, \quad f(2,\pm 1) = 2\cdot 2 + 5\cdot(\pm 1)^2 = 9,$$

we see that the maximum value is 9 and the minimum value is -6.

Comment. This problem illustrates that solving Lagrange multiplier equations sometimes requires discipline and attention to detail.

Here is slightly different problem.

$\boxed{\mathbf{12}}$ (January 2010) Find the point on the plane $2x - y + 2z = 16$ that is nearest the origin.

Solution. This problem initially looks confusing since it does not state explicitly the function to be minimized. The key phrase is "nearest the origin", which means minimize the distance between the $(0, 0, 0)$ and a point (x, y, z) on the plane. Hence the function to minimize is

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

But minimizing a square root is equivalent to minimizing the quantity under the square root symbol, namely $x^2 + y^2 + z^2$.

Hence we need to minimize $x^2 + y^2 + z^2$ subject to the constraint $2x - y + 2z = 16$. The respective gradients are $\langle 2x, 2y, 2z\rangle$ and $\langle 2, -1, 2\rangle$. So we need to solve

$$2x = \lambda \cdot 2, \quad 2y = \lambda \cdot (-1), \quad 2z = \lambda \cdot 2, \; 2x - y + 2z = 16.$$

The first three equations imply $x = \lambda$, $y = -\frac{1}{2}\lambda$ and $z = \lambda$. Substituting into the constraint gives

$$2\lambda - (-\tfrac{1}{2}\lambda) + 2\lambda = 16 \;\Rightarrow\; \tfrac{9}{2}\lambda = 16 \;\Rightarrow\; \lambda = \tfrac{2}{9} \cdot 16 = \tfrac{32}{9}.$$

Hence $x = \frac{32}{9}$, $y = -\frac{1}{2} \cdot \frac{32}{9} = -\frac{16}{9}$, and $z = \frac{32}{9}$. This unique point must be point on the plane closest to the origin, which gives the minimum distance

$$\sqrt{\left(\tfrac{32}{9}\right)^2 + \left(-\tfrac{16}{9}\right)^2 + \left(\tfrac{32}{9}\right)^2} = \sqrt{\left(\tfrac{16}{9}\right)^2(4 + 1 + 4)} = \tfrac{16}{9} \cdot \sqrt{9} = \tfrac{16}{3}.$$

Comment. The last line used the common factor $(\frac{16}{9})^2$ to compute a complicated looking square root. This is a good illustration of why algebra is such a powerful tool in mathematics.

Here is a problem for you to do.

$\boxed{\mathbf{13}}$ (March 2016) Find the absolute maximum value of the function

$$f(x, y) = x - 2y + 3$$

on the domain D given by the circle

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 5\}.$$

Answer: 8

Absolute Minima and Maxima. A problem may ask for the maximum and maximum values (also called extreme values) of a differentiable function $f(x, y)$ on a closed and bounded region in the plane. Extreme values are known to exist in this situation. They can occur in one of two places:

- In the interior of the region, where they occur among the critical points of f.
- On the boundary of the region, where you use Lagrange multipliers. The constraint is the defining equation of the boundary.

Note that when you find the critical points in the interior, you do *not* need to apply the second derivative test (which would only tell you about local maxima or minima). Here is a typical problem.

$\boxed{14}$ (January 2016) Find the points at which the absolute maximum and minimum of the function $f(x, y) = xy - 1$ on the disk $x^2 + y^2 \leq 2$ occur. State all points where the extrema occur as well as the maximum and minimum values.

Solution. The first step is to find the critical points of f in the interior of the disk. This is easy, since $f_x = y = 0$ and $f_y = x = 0$ imply that $(x, y) = (0, 0)$.

We next use Lagrange multipliers to find where the maximum or minimum values of f can occur on the boundary $x^2 + y^2 = 2$. Writing this as $g(x, y) = x^2 + y^2 - 2 = 0$, Lagrange multipliers gives the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0,$$

where can be written as

$$y = \lambda \cdot 2x, \quad x = \lambda \cdot 2y, \quad x^2 + y^2 - 2 = 0.$$

The first two equations imply $y = 4\lambda^2 y$, so $y(4\lambda^2 - 1) = 0$. So there are two cases to pursue:

$y = 0$: This implies $x = \lambda \cdot 2 \cdot 0 = 0$, which doesn't satisfy $g(x, y) = 0$. So no solutions here.

$4\lambda^2 - 1 = 0$: This implies $2\lambda = \pm 1$. It follows that $y = \pm x$. Substituting into the constraint gives $2x^2 = 2$, so $x = \pm 1$. Hence we get the four boundary points $(\pm 1, \pm 1)$.

Thus there are five points on the disk where extrema could possibly occur: $(0, 0)$, $(\pm 1, \pm 1)$. Since

$$f(0, 0) = -1, \quad f(1, 1) = f(-1, -1) = 0, \quad f(1, -1) = f(-1, 1) = -2,$$

we conclude that the absolute maximum occurs at $(1, 1)$ and $(-1, -1)$ and has a value of 0, while the absolute minimum occurs at $(1, -1)$ and $(-1, 1)$ and has a value of -2.

Here is a problem that combines several types of questions about maxima and minima.

$\boxed{15}$ (March 2007) Consider $f(x, y) = 2x^2 + 3y^2$ on the closed disk $x^2 + y^2 \leq 1$.

(a) Find the critical points of f in the interior of the disk and classify them using the 2nd derivative test. ANSWER: local minimum at $(0, 0)$

(b) Find the minimum and maximum values of $f(x, y)$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange multipliers. ANSWER: minimum value 2, maximum value 3

(c) What are the minimum and maximum values of $f(x, y)$ on $x^2 + y^2 \leq 1$?
ANSWER: minimum value 0, maximum value 3

4 Double Integrals

Given a function $f(x, y)$ on a region R in the plane, one can define the double integral $\iint_R f(x, y)\, dA$.

Iterated Integrals. When the region R has a nice description in Cartesian coordinates, the double integral can be expressed as an iterated integral in two ways:

- The first way is

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

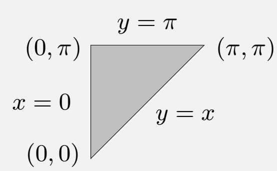
when R consists of all points (x, y) where $a \leq x \leq b$, and for x in this range, $g_1(x) \leq y \leq g_2(x)$. So $y = g_2(x)$ is the top of R, $y = g_1(x)$ is the bottom, and $x = a$, $x = b$ are the sides. When doing the inner integral $\displaystyle\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$, you should treat x as a constant.

- The second way is
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$
when R consists of all points (x,y) where $c \le y \le d$, and for y in this range, $h_1(y) \le x \le h_2(y)$. From the point of view of someone on the y-axis, $x = h_2(y)$ is the "top" of R, $x = h_1(y)$ is the "bottom", and $y = c$, $y = d$ are the "sides". When doing the inner integral $\displaystyle\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx$, treat y as a constant.

Some double integrals can be expressed as iterated integrals in both ways. Here is an example.

$\boxed{\textbf{16}}$ (March 2014) Evaluate $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\,dy\,dx$.

Solution. The inner integral $\displaystyle\int_x^\pi \frac{\sin(y)}{y}\,dy$ is impossible by the standard techniques of integration. Because of this, we change the order of integration. To do so, the first step is to understand region of integration, which is the following triangle:



Be sure you know how the limits of integration give this triangle. Then changing the order of integration gives
$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\,dy\,dx = \int_0^\pi \int_0^y \frac{\sin(y)}{y}\,dx\,dy = \int_0^\pi \left(\frac{\sin(y)}{y}\,x\Big|_{x=0}^{y} \right) dy = \int_0^\pi \frac{\sin(y)}{y}\,y\,dy$$
$$= \int_0^\pi \sin(y)\,dy = -\cos(y)\Big|_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

Be sure you know how to figure out the new limits of integration.

This solution requires that you remember some basic calculus, including the integral of $\sin(y)$ and the values of trig functions such as $\cos(\pi)$ and $\cos(0)$. Here is a similar problem you should do yourself.

$\boxed{\textbf{17}}$ (March 2010) Evaluate $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$. Answer: $\frac{52}{9}$

Comments. This problem has two new features:

- The region of integration has $x = \sqrt{y}$ as one of its boundary curves. When you change of the order of integration, the inverse function $y = x^2$ appears:
$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy = \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx.$$

It is *essential* that you be able to draw the region of integration and see how the limits change when you change the order of integration.

- Another step in the solution involves recognizing that $\int_0^2 x^2\sqrt{x^3+1}\,dx$ can be done by the substitution method (also called u-substitution). Again this is part of basic calculus that you need to know for the exam.

Polar Coordinates. Be familiar with polar coordinates (r,θ) in the plane and how to convert between Cartesian and polar coordinates. When the region R in a double integral $\iint_R f(x,y)\,dA$ has a nice description in polar coordinates, the integral can be expressed as the iterated integral

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\,r\,dr\,d\theta,$$

where R consists of all points with polar coordinates (r,θ) such that $\alpha \le \theta \le \beta$, and for θ in this range, $h_1(\theta) \le r \le h_2(\theta)$. When doing the inner integral $\int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\,r\,dr$, you should treat θ as a constant.

$\boxed{\mathbf{18}}$ (January 2013) Evaluate $\iint_R (x+y)\,dy\,dx$, where R is the top half of the circle of radius 2 centered at the origin.

Solution. The region R is a semicircle of radius 2 with polar description $0 \le \theta \le \pi$ and $0 \le r \le 2$. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we obtain

$$\begin{aligned}
\iint_R (x+y)\,dy\,dx &= \int_0^\pi \int_0^2 (r\cos(\theta) + r\sin(\theta))\,r\,dr\,d\theta \\
&= \int_0^\pi \int_0^2 (\cos(\theta)+\sin(\theta))\,r^2\,dr\,d\theta \\
&= \int_0^\pi \left((\cos(\theta)+\sin(\theta))\,\frac{r^3}{3}\bigg|_{r=0}^{2} \right) d\theta \\
&= \frac{8}{3}\int_0^\pi \cos(\theta)+\sin(\theta)\,d\theta = \frac{8}{3}\big(\sin(\theta)-\cos(\theta)\big)\Big|_0^\pi \\
&= \frac{8}{3}\big(\sin(\pi)-\cos(\pi)\big) - \frac{8}{3}\big(\sin(0)-\cos(0)\big) \\
&= \frac{8}{3}\big(0-(-1)\big) - \frac{8}{3}\big(0-1\big) = \frac{16}{3}.
\end{aligned}$$

Comment. In Cartesian coordinates, $\iint_R (x+y)\,dy\,dx = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx$. For practice, you should do this integral, which will require a u-substitution. Be aware that given a double integral such as $\iint_R (x+y)\,dy\,dx$, you may be asked to express it as an iterated integral in both Cartesian and polar coordinates.

The above problem is good reminder that you need to be able to convert between Cartesian and polar coordinates. Also remember that for double integrals, $dA = dx\,dy$ or $dy\,dx$ in Cartesian coordinates and $dA = r\,dr\,d\theta$ in polar coordinates.

Finding Area and Volume. The basic area interpretation of the double integral is $\iint_R 1\,dA = \text{Area}(R)$. See $\boxed{25}$ for a problem that uses this. For volumes, there are two situations to consider:

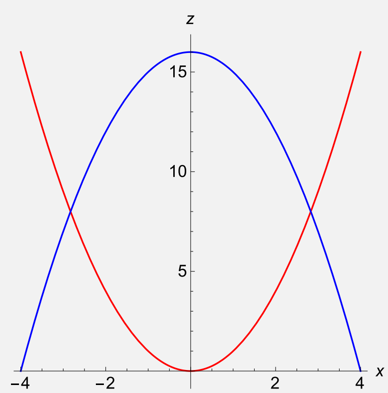
- When $f(x,y) \geq 0$ on R, $\iint_R f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ for $(x,y) \in R$.
- More generally, suppose that a 3-dimensional region V in $\mathbb{R}^3$ consists of all points (x,y,z) such that $(x,y) \in R$ and $f_1(x,y) \leq z \leq f_2(x,y)$. Thus $z = f_2(x,y)$ is the top of V, $z = f_1(x,y)$ is the bottom, and the sides lie over the boundary of R. In this case, the volume of V is

$$\text{Vol}(V) = \iint_R \big(f_2(x,y) - f_1(x,y)\big)\,dA.$$

Here is an example.

$\boxed{19}$ (March 2014) Find the volume of the region bounded by the two paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Solution. The paraboloid $z = x^2 + y^2$ starts at the origin and opens up, while $z = 16 - x^2 - y^2$ starts at $(0,0,16)$ and opens down. Visualizing this in 3-dimensions is not easy. However, the graphs are symmetric about the z-axis, which means that the cross-sections where $y = 0$ give useful information. The cross-sections are $z = x^2$ and $z = 16 - x^2$, which are easy to draw in the (x,z)-plane:



Be sure you understand the importance of pictures like this. The 3-dimensional region we want is trapped between the two surfaces. The top is $z = 16 - x^2 - y^2$ and the bottom is $z = x^2 + y^2$. To find the region R in the plane, we consider where the top and bottom meet, which is where

$$16 - x^2 - y^2 = x^2 + y^2 \;\Leftrightarrow\; 16 = 2(x^2 + y^2) \;\Leftrightarrow\; x^2 + y^2 = 8.$$

Thus R is the region where $x^2 + y^2 \leq 8$, the circle of radius $\sqrt{8}$ centered at the origin. We compute:

$$\begin{aligned}
\text{volume} &= \iint_R (16 - x^2 - y^2) - (x^2 + y^2)\,dA = \iint_R 16 - 2(x^2 + y^2)\,dA \\
&= \int_0^{2\pi}\int_0^{\sqrt{8}} (16 - 2r^2)\,r\,dr\,d\theta = 2\pi \int_0^{\sqrt{8}} (16 - 2r^2)\,r\,dr,
\end{aligned}$$

where the last equality follows since the inner integral is independent of θ and $\int_0^{2\pi} d\theta = 2\pi$. We now continue to the final answer:

$$= 2\pi \int_0^{\sqrt{8}} 16r - 2r^3\,dr = 2\pi\left(16\frac{r^2}{2} - 2\frac{r^4}{4}\right)\Big|_0^{\sqrt{8}}$$
$$= 2\pi\left(16\cdot\frac{8}{2} - 2\cdot\frac{64}{4}\right) = 2\pi(64-32) = 2\pi\cdot 32 = 64\pi.$$

Comment. Polar coordinates work nicely because the region R has a nice polar description and the function $16 - 2(x^2+y^2)$ converts nicely to polar coordinates. This problem also can be done using a triple integral. Do you see why cylindrical coordinates would be the best choice among the possible options for triple integral coordinate systems?

5 Triple Integrals

Given a function $f(x,y,z)$ on a region R in $\mathbb{R}^3$, one can define the triple integral $\iiint_R f(x,y)\,dV$.

Cartesian, Cylindrical and Spherical Coordinates. You need to be able to work with triple integrals in three coordinate systems:

- Cartesian coordinates x,y,z, where $dV = dx\,dy\,dz$ or $dz\,dy\,dx$. Other orders are possible.
- Cylindrical coordinates r,θ,z, where $dV = r\,dz\,dr\,d\theta$.
- Spherical coordinates ρ,ϕ,θ, where $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Be sure you know the geometric meaning of these coordinate systems and how to convert between them.

Here is a triple integral problem that uses gradients and lengths of vectors.

$\boxed{\mathbf{20}}$ (March 2006) Let $F(x,y,z) = x^2+y^2+z^2$.

(a) Compute the gradient vector ∇F.

(b) Compute $\iiint_R ||\nabla F||\,dV$, where R is the region $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 1\}$ and $||v||$ denotes the length of the vector v.

Solution. (a) $\nabla F = \frac{\partial}{\partial x}(x^2+y^2+z^2)\,\mathbf{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)\,\mathbf{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)\,\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

(b) $||\nabla F|| = ||2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}|| = \sqrt{(2x)^2+(2y)^2+(2z)^2} = 2\sqrt{x^2+y^2+z^2}$. Hence

$$\iiint_R ||\nabla F||\,dV = \iiint_R 2\sqrt{x^2+y^2+z^2}\,dV.$$

The region R (a sphere of radius 1) and function $2\sqrt{x^2+y^2+z^2}$ suggest spherical coordinates. Then

$$\iiint_R 2\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 2\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\int_0^1 2\rho^3\sin\phi\,d\rho\,d\phi$$
$$= 4\pi\int_0^{\pi}\left(\frac{\rho^4}{4}\sin\phi\right)\Big|_{\rho=0}^1 d\phi = \pi\int_0^{\pi}\sin\phi\,d\phi$$
$$= \pi\big(-\cos\phi\big)\Big|_0^{\pi} = \pi\big(-\cos\pi - (-\cos 0)\big) = \pi(-(-1)+1) = 2\pi.$$

Comment. Be sure you understand the limits of integration.

Finding Volume. The basic volume interpretation of the triple integral is $\iiint_R 1\,dV = \text{Vol}(R)$. You may be asked to express a volume in all three coordinate systems and evaluate one of them. Here is an example.

$\boxed{\textbf{21}}$ (January 2011) Let V be the region in $\mathbb{R}^3$ inside the sphere $x^2+y^2+z^2=1$ and above the plane $z=0$.

(a) Express the volume of V in cartesian, cylindrical and spherical coordinates.
(b) Evaluate one of the integrals found in part (a).

Solution. (a) We are working with a hemisphere whose projection onto the xy-plane is $x^2+y^2 \le 1$. The answer for cartesian coordinates is:

$$\text{Vol}(V) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx.$$

For cylindrical coordinates, the region in the xy-plane is described by $0 \le r \le 1$, with no restriction on θ. The top half of the sphere is $z = \sqrt{1-r^2}$, so the integral becomes:

$$\text{Vol}(V) = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta.$$

For spherical coordinates, the hemisphere has radius 1, so $0 \le \rho \le 1$. There is no restriction on θ, and being above the plane $z=0$ means that $0 \le \phi \le \pi/2$. Therefore the integral is:

$$\text{Vol}(V) = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

(b) It makes sense to use cylindrical coordinates or spherical coordinates since they have simpler limits of integration. Here are solutions for both.

For cylindrical coordinates:

$$\begin{aligned}
\text{Vol}(V) &= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 r\sqrt{1-r^2}\,dr\,d\theta \\
&= 2\pi \int_0^1 r\sqrt{1-r^2}\,dr \quad u = 1-r^2,\ du = -2r\,dr,\ \text{so}\ -\frac{1}{2}\,du = r\,dr \\
&= 2\pi \int_1^0 \sqrt{u}\left(-\frac{1}{2}du\right) = 2\pi \cdot \frac{1}{2} \int_0^1 u^{1/2}\,du\,d\theta = \pi \cdot \frac{2}{3} u^{3/2}\Big|_0^1 = \frac{2\pi}{3}.
\end{aligned}$$

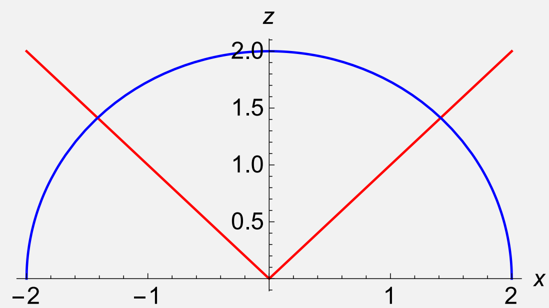
For spherical coordinates:

$$\begin{aligned}
\text{Vol}(V) &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 2\pi \int_0^{\pi/2} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi \\
&= 2\pi \int_0^{\pi/2} \frac{1}{3}\rho^3 \sin\phi\Big|_{\rho=0}^1\,d\phi = 2\pi \int_0^{\pi/2} \frac{1}{3}\sin\phi\,d\phi \\
&= \frac{2\pi}{3}\big(-\cos\phi\big)\Big|_0^{\pi/2} = \frac{2\pi}{3}\Big(-\cos\frac{\pi}{2} - (-\cos 0)\Big) = \frac{2\pi}{3}.
\end{aligned}$$

Sometimes you are simply asked for the volume, leaving it to you to pick the best coordinate system.

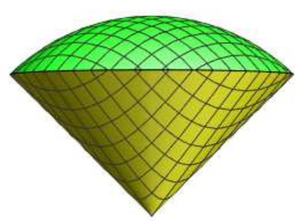
$\boxed{\textbf{22}}$ (January 2008) Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$. ANSWER: $\frac{16\pi}{3}(1 - \frac{\sqrt{2}}{2})$

Comment. Be sure you understand why drawing a picture is the best place to start. Similar to $\boxed{\textbf{19}}$, the symmetry about the z-axis means that the cross-section with $y = 0$ gives you good information. This means graphing $x^2 + z^2 = 4$ and $z = \sqrt{x^2} = |x|$:



This problem can be done in either cylindrical or spherical coordinates, though one of these has very simple limits of integration. Do you see how above picture implies that $0 \leq \phi \leq \pi/4$ when you use spherical coordinates?

Here is a 3-dimensional picture of the region in $\boxed{\textbf{22}}$:



If you can draw something like this, great, but keep in mind that if you understand how cross-sections work, it is often not essential to make a 3-dimensional drawing.

6 Line Integrals of Vector Fields

You need to know line integrals in two and three dimensions:

$$\text{in } \mathbb{R}^2 : \int_C f(x,y)\,dx + g(x,y)\,dy,\ C \text{ a curve in } \mathbb{R}^2$$

$$\text{in } \mathbb{R}^3 : \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz,\ C \text{ a curve in } \mathbb{R}^3.$$

These line integrals are sometimes written $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is a vector field. More precisely,

$$\text{in } \mathbb{R}^2 : \int_C f(x,y)\,dx + g(x,y)\,dy = \int_C \mathbf{F} \cdot d\mathbf{r},\ \mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j},\ d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}.$$

$$\text{in } \mathbb{R}^3 : \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz = \int_C \mathbf{F} \cdot d\mathbf{r},\ \mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k},\ d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}.$$

Also know how to compute a line integral. If C is a curve in the plane is parametrized by $(x(t), y(t))$ for $a \leq t \leq b$, then

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_a^b \Big(f(x(t),y(t))x'(t) + g(x(t),y(t))y'(t) \Big)\,dt.$$

See $\boxed{\mathbf{24}}$ for a problem that uses this formula. There is a similar formula in three dimensions.

Multivariable calculus courses also discuss line integrals of the form $\int_C f\,ds$. You do not need to know this type of line integral for the Comprehensive Exam.

Fundamental Theorem of Calculus for Line Integrals. Suppose that a curve C in $\mathbb{R}^2$ or $\mathbb{R}^3$ starts at a point p and ends at a point q. Given a differentiable function F with continuous partials on a region containing C, we have

$$\int_C \nabla F \cdot d\mathbf{r} = F(q) - F(p).$$

This implies that we get the same answer no matter which path we take to get from p to q. This is called independence of path. Here is a problem about independence of path that recalls a useful technique.

$\boxed{\mathbf{23}}$ (January 2017) Show that the line integral

$$\int_C z^2\,dx + 2y\,dy + 2xz\,dz$$

depends only on the endpoints of the path C and not on the path taken between those endpoints.

Solution. Since $\displaystyle\int_C \nabla F \cdot d\mathbf{r}$ is independent of path, the strategy will be to find F such that

$$\nabla F = \frac{\partial F}{\partial x}\,\mathbf{i} + \frac{\partial F}{\partial y}\,\mathbf{j} + \frac{\partial F}{\partial z}\,\mathbf{k} = z^2\,\mathbf{i} + 2y\,\mathbf{j} + 2xz\,\mathbf{k}.$$

We find F as follows. First,

$$\frac{\partial F}{\partial x} = z^2 \;\Rightarrow\; F = xz^2 + G(y,z),$$

and then

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\big(xz^2 + G(y,z)\big) = 2y \;\Rightarrow\; \frac{\partial G(y,z)}{\partial y} = 2y \;\Rightarrow\; G = y^2 + H(z) \;\Rightarrow\; F = xz^2 + y^2 + H(z).$$

The final equation to consider is

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}\big(xz^2 + y^2 + H(z)\big) = 2xz \;\Rightarrow\; 2xz + H'(z) = 2xz.$$

If we pick $H(z) = 0$, then $F(x,y,z) = xz^2 + y^2$ has the property that $\nabla F = z^2\,\mathbf{i} + 2y\,\mathbf{j} + 2xz\,\mathbf{k}$. It follows that $\displaystyle\int_C z^2\,dx + 2y\,dy + 2xz\,dz = \int_C \nabla F \cdot d\mathbf{r}$ is independent of the path.

Green's Theorem. The basic version of Green's Theorem says that if a simple closed curve C in the plane encloses a region R and is oriented counterclockwise, then

$$\int_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

Problems involving Green's Theorem occur frequently on the multivariable exam. Here are two examples.

$\boxed{\mathbf{24}}$ (January 2015) Let C be the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$, oriented counterclockwise, and let $\mathbf{F}(x,y) = \langle xy, x^2\rangle$.

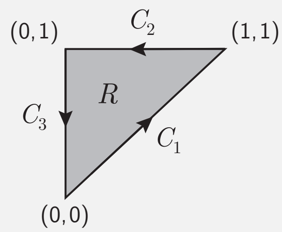
(a) According to Green's Theorem, the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C xy\,dx + x^2\,dy$$

is equal to a certain double integral. Set up this double integral.

(b) Verify Green's theorem in this case by evaluating both the line integral and the double integral in part (a).

Solution. (a) $\displaystyle\int_C xy\,dx + x^2\,dy = \iint_R \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\,dA = \iint_R 2x - x\,dA = \iint_R x\,dA$, where R is the region enclosed by C:



In this picture, the boundary of R is $C = C_1 + C_2 + C_3$.

(b) Expressing the double integral as an iterated integral, we obtain

$$\iint_R x\,dA = \int_0^1\int_x^1 x\,dy\,dx = \int_0^1 (xy)\Big|_{y=x}^{1}\,dx = \int_0^1 x - x^2\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

To compute the line integral, we parametrize C_1, C_2, C_3 via

$$\begin{aligned} C_1 &: (x,y) = (t,t),\ 0 \le t \le 1\\ C_2 &: (x,y) = (1-t,1),\ 0 \le t \le 1\\ C_3 &: (x,y) = (0,1-t),\ 0 \le t \le 1. \end{aligned}$$

Be sure you understand how these parametrizations were obtained (remember that C is oriented counterclockwise). Then:

$$\begin{aligned}
\int_{C_1} xy\,dx + x^2\,dy &= \int_0^1 \left(t\cdot t\cdot 1 + t^2\cdot 1\right)dt = \int_0^1 2t^2\,dt = \frac{2}{3}t^3\Big|_0^1 = \frac{2}{3}\\
\int_{C_2} xy\,dx + x^2\,dy &= \int_0^1 \left((1-t)\cdot 1\cdot(-1) + (1-t)^2\cdot 0\right)dt = \int_0^1 t-1\,dt = \left(\frac{1}{2}t^2 - t\right)\Big|_0^1 = -\frac{1}{2}\\
\int_{C_3} xy\,dx + x^2\,dy &= \int_0^1 \left(0\cdot(1-t)\cdot 0 + 0^2\cdot(-1)\right)dt = \int_0^1 0\,dt = 0.
\end{aligned}$$

This gives

$$\int_C xy\,dx + y^2\,dy = \int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy + \int_{C_3} xy\,dx + x^2\,dy = \frac{2}{3} - \frac{1}{2} + 0 = \frac{1}{6},$$

which agrees with the double integral computed above.

Comment. Be prepared to compute a line integral via a parametrization. On the other hand, if you aren't specifically required to do the line integral that way, consider other options instead. If it's possible to use either Green's Theorem (if C is a closed loop) or the Fundamental Theorem for Line Integrals (if $\mathbf{F} = \nabla f$ for some f), it may be better to do the integral that way, instead.

$\boxed{\textbf{25}}$ (March 2011) Suppose that C is a closed curve, oriented counterclockwise, and C encloses a region R whose area is 5. Find $\displaystyle\int_C (x^2y^3 - 3y)\,dx + x^3y^2\,dy$.

Solution. We don't know C but we know something about the region R it encloses. Hence it makes sense to use Green's Theorem. We compute:

$$\begin{aligned}\int_C (x^2y^3 - 3y)\,dx + x^3y^2\,dy &= \iint_R \frac{\partial}{\partial x}(x^3y^2) - \frac{\partial}{\partial y}(x^2y^3 - 3y)\,dA \\ &= \iint_R 3x^2y^2 - (3x^2y^2 - 3)\,dA = \iint_R 3\,dA \\ &= 3\iint_R 1\,dA = 3\,\text{Area}(R) = 3 \cdot 5 = 15.\end{aligned}$$

Comment. Note the use of the area property of double integrals: $\displaystyle\iint_R 1\,dA = \text{Area}(R)$.

$\boxed{\textbf{26}}$ (January 2010) Evaluate $\displaystyle\int_C \cos(x^2)\,dx + (3xy^2 + x^3)\,dy$, where C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Solution. Let R be the region enclosed by C, which is the disk of radius 2 centered at the origin. Using Green's Theorem,

$$\int_C \cos(x^2)\,dx + (3xy^2 + x^3)\,dy = \iint_R \frac{\partial}{\partial x}(3xy^2 + x^3) - \frac{\partial}{\partial y}(\cos(x^2))\,dA = \iint_R (3y^2 + 3x^2)\,dA$$

Let's use polar coordinates to compute this double integral: so $(3x^2 + 3y^2) = 3r^2$ and $dA = r\,dr\,d\theta$. The integral becomes

$$\int_0^{2\pi}\int_0^2 3r^3\,dr\,d\theta = \int_0^{2\pi} \frac{3}{4}r^4\Big|_{r=0}^2\,d\theta = \int_0^{2\pi} 12\,d\theta = 12\theta\Big|_0^{2\pi} = 24\pi.$$

Comment. We *could* have parametrized the curve C by $\mathbf{r}(t) = (2\cos t, 2\sin t)$ for $0 \le t \le 2\pi$, but that would have led to a horrible mess. So we turned to Green's Theorem, which we could use because C is a closed curve, and that made things simpler. Also, we did the double integral using polar coordinates rather than $dx\,dy$, again because it made things simpler. *Moral*: If you have multiple options, choose the simple one.

Here is another problem you should do for practice.

$\boxed{\textbf{27}}$ (January 2013) Evaluate $\displaystyle\int_C (2xy + \tan(x^3))\,dx + (x^2 + 2xy)\,dy$, where C is the closed curve that begins at $(0,0)$, then follows $y = x^2$ to $(1,1)$, next follows $y = 1$ to $(0,1)$, and finally follows $x = 0$ back to $(0,0)$. Answer: $\frac{4}{5}$

7 Preparing for the Multivariable Exam

Now that you have finished reading the content part of the Study Guide, what should you do next to prepare for the exam? The key thing to keep in mind is that

You need an active knowledge of multivariable calculus.

Here are a some suggestions to help you achieve this.

Read the Study Guide Actively. There are many places where the Study Guide asks you to do a problem. Do so. For those problems, the Study Guide gives the final answer so you can check your work. However, on the exam, we grade *all* of your work, not just the final answer.

Read Your Notes and Your Multivariable Calculus Book. In many places in the Study Guide, we say "Know . . . ", without stating the facts precisely. This is deliberate, since we want you to refer to your notes and your calculus book when studying for the exam.

If you do not have a copy of your multivariable calculus book, note that *Multivariable Calculus*, 7th edition, by James Stewart is used frequently in Math 211. The Amherst College Library has a copy, and copies are also available in the QCenter.

Not everything covered in your multivariable calculus course is part of the exam. For example, the chain rule for multivariable functions is an important topic but is not on the exam. This Study Guide and the *Syllabus for Comprehensive Examination in Multivariable Calculus* (*Math 211*) list the topics that you need to know for the exam.

Study Old Exams and Solutions. The Department website has a collection of old Comprehensive Exams, many with solutions. It is very important to do practice problems. This is one of the key ways to acquire an active knowledge of multivariable calculus. However, there are two dangers to be aware of when using old exams and solutions:

- Thinking that the exams tell you what to study. Every topic on the *Syllabus* and in this Study Guide is fair game for an exam question.
- Reading the solutions. This is passive. To get an active knowledge of the material, do problems from the old exams yourself, and *then* check the solutions. The more you can do this, the better.

Work Together, Ask Questions, and Get Help. Studying with your fellow math majors can help. You can learn a lot from each other. Faculty are delighted to help. Don't hesitate to ask us questions and show us your solutions so we can give you feedback. The QCenter has excellent people who have helped many students prepare for the Comprehensive Exam.

Start Now. Properly preparing for the Comprehensive Exam will take longer than you think. Start now.