Walker Prize Examination 2019

April 8, 2019

1. *Find the minimum possible value of* $x^2 + y^2$ *where* x, y *are positive real numbers such that*

$$xy(x^2 - y^2) = x^2 + y^2.$$

Minimum value is $\boxed{4}$

2. *Compute* $\displaystyle\sum_{n=2}^{\infty} \frac{(n-2)! + n}{n!}$ *where* $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.

The sum is $\boxed{e}$

3. *A shipping clerk has five boxes of different but unknown weights. All of the boxes weigh less than 100 pounds, but, unfortunately, the only scale available only reads weights over 100 pounds. The clerk decides to weigh the boxes in pairs so that each box is weighed with every other box. The weights of all possible pairs are 110, 112, 113, 114, 115, 116, 117, 118, 120, and 121 pounds. Determine the weights of the five boxes.*

The weights are: $\boxed{54, 56, 58, 59, 62}$ pounds.

4. *Find a real number* c *such that the parabolas* $y = -x^2$ *and* $y = 2x^2 + c$ *have two common tangent lines that meet at a right angle.*

c should be $\boxed{3/8}$

5. *Find the two smallest prime factors of* $3^{2^{2019}} - 2^{2^{2019}}$.

Clearly 2 and 3 are not factors. Repeatedly using difference of two squares shows that $5 = 2 + 3$ and $13 = 2^2 + 3^2$ are factors. To rule out 7 and 11, show that the sequences 3^{2^n} and 2^{2^n} are periodic and never line up mod 7 or mod 11.

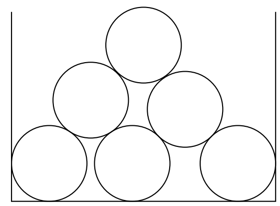
The two smallest prime factors are $\boxed{5, 13}$

6. *Your radio needs two good batteries to work. You have five batteries, and you know that three of them are good, but you don't know which. What is the smallest number of trials you would need in order to be certain to find two good batteries?*

(Each trial consists of picking two batteries of the five and trying them in the radio. If the radio comes on, you know both batteries are good. If the radio does not come on, you know at least one of those two batteries is bad. You could of course just try all ten combinations of two batteries from the five, but is there a better strategy?)

After $\boxed{\text{3 trials}}$ you can be sure that 2 particular batteries are good.

7. *Six identical bottles of wine are stacked horizontally in a bin as in the following diagram. Prove that the top bottle has its center directly over the center of the bin (even when the middle bottle of the bottom row does not).*



8. *Find all positive integers a, b, c such that*

$$a!\, b! = a! + b! + c!$$

The only solution is $\boxed{a = 3, b = 3, c = 4}$

9. *What is the largest number of points you can put inside a square of side length 2 units in such a way that the distance between any pair of the points is more than (and* not *equal to) 1 unit?*

The largest number of points possible is $\boxed{8}$

10. *Suppose $f : [0, \infty) \to \mathbb{R}$ is defined by*

$$f(a) = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2}\, dx.$$

Calculate $f(2019)$.

$f(2019)$ is equal to $\boxed{\pi e^{-2019}}$